How does volume and surface area affect a skydiver's falling speed?
How fast a skydiver falls depends on both the volume and the cross-sectional surface area of the skydiver. Different falling positions result in different ratios of volume to cross-sectional surface area: the larger the ratio, the greater the skydiver's falling speed.

**Think & Discuss**

The table below gives the volume and cross-sectional surface area for a 70 inch tall skydiver in each of three different positions.

<table>
<thead>
<tr>
<th>Position</th>
<th>Volume (in.(^3))</th>
<th>Cross-sectional surface area (in.(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Face-to-Earth</td>
<td>13,000</td>
<td>1300</td>
</tr>
<tr>
<td>Sitting</td>
<td>13,000</td>
<td>800</td>
</tr>
<tr>
<td>Headfirst</td>
<td>13,000</td>
<td>350</td>
</tr>
</tbody>
</table>

1. Find the ratio of volume to cross-sectional surface area for the skydiver in each of the three positions.
2. In which position will the skydiver have the greatest falling speed? Explain.

**Learn More About It**

You will use a geometric model to write a rational expression for the ratio of a skydiver's volume to his or her cross-sectional surface area in Example 8 on p. 557 and in Ex. 15 on p. 558.

APPLICATION LINK Visit www.mcdougallittell.com for more information on skydiving.
Chapter 9 is about rational expressions, functions, and equations. In Chapter 9 you’ll learn

- how to simplify and perform operations with rational expressions.
- how to graph rational functions and solve rational equations.
- how to use variation models and rational models in real-life situations.

**KEY VOCABULARY**

- rational numbers, p. 3
- x-intercept, p. 84
- direct variation, p. 94
- zero of a function, p. 259
- degree of a polynomial function, p. 329
- asymptote, p. 465
- inverse variation, p. 534
- joint variation, p. 536
- rational function, p. 540
- hyperbola, p. 540
- simplified form of a rational expression, p. 554
- complex fraction, p. 564

**Are you ready for the chapter?**

**SKILL REVIEW** Do these exercises to review key skills that you’ll apply in this chapter. See the given reference page if there is something you don’t understand.

The variables $x$ and $y$ vary directly. Write an equation that relates the variables. (Review Example 6, p. 94)

1. $x = 2, y = 5$
2. $x = 1, y = 0.1$
3. $x = 8, y = -2$
4. $x = -3, y = 12$

Multiply the polynomials. (Review Example 5, p. 13; Example 4, p. 339)

5. $5(3x - 1)$
6. $(x - 1)(x + 4)^2$
7. $-x(x^2 - 5)$
8. $x(x - 1)(x + 8)$

Factor the polynomial. (Review Examples 1–4, pp. 256 and 257; Examples 1–3, p. 346)

9. $x^2 - 6x + 9$
10. $4x^3 - 4$
11. $8x^3 - 162x$
12. $6x^2 + 7x - 5$

Find all the real zeros of the function. (Review Example 7, p. 259; Example 4, p. 345; Example 1, p. 359)

13. $y = x^2 + 2x$
14. $y = x^2 + 2x - 15$
15. $y = x^3 - 2x^2 - 7x - 4$

**Dictionary of Functions**

Make a dictionary of all the types of functions you have learned in this course. For each entry, include the general form of the function and an example of the function and its graph. Continue to add entries as you work through this chapter. Use your dictionary as a study and reference tool.
Investigating Inverse Variation

QUESTION What is the relationship between the distance you are standing from your partner and the apparent height of your partner?

EXPLORING THE CONCEPT

1. Have your partner stand with his or her back against a wall. Place the end of a tape measure against the wall and between your partner’s feet. Use masking tape to mark off distances of 3 meters, 4 meters, . . . , 9 meters from the wall.

2. Stand facing your partner, with your toes just touching the 3 meter mark. Hold a centimeter ruler at arm’s length and line up the “0” end of the ruler with the top of your partner’s head. Measure (to the nearest centimeter) the apparent height of your partner at this distance.

3. Repeat Step 2 for each of the marked distances and record your results in a table like the one shown.

<table>
<thead>
<tr>
<th>Distance (m)</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
</table>

DRAWING CONCLUSIONS

1. Does apparent height vary directly with distance? Justify your answer mathematically.

2. Multiply the paired values of distance and apparent height together. What do you notice?

3. Based on your results from Exercise 2, write an equation relating distance and apparent height.

4. Use your equation from Exercise 3 to predict your partner’s apparent height at a distance not listed in your table. Test your prediction by standing that distance from your partner and measuring his or her apparent height. How close was your prediction?
In Lesson 2.4 you learned that two variables $x$ and $y$ show direct variation if $y = kx$ for some nonzero constant $k$. Another type of variation is called inverse variation. Two variables $x$ and $y$ show inverse variation if they are related as follows:

$$y = \frac{k}{x}, \quad k \neq 0$$

The nonzero constant $k$ is called the constant of variation, and $y$ is said to vary inversely with $x$.

### Example 1: Classifying Direct and Inverse Variation

Tell whether $x$ and $y$ show direct variation, inverse variation, or neither.

<table>
<thead>
<tr>
<th>GIVEN EQUATION</th>
<th>REWRITTEN EQUATION</th>
<th>TYPE OF VARIATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $\frac{y}{5} = x$</td>
<td>$y = 5x$</td>
<td>Direct</td>
</tr>
<tr>
<td>b. $y = x + 2$</td>
<td></td>
<td>Neither</td>
</tr>
<tr>
<td>c. $xy = 4$</td>
<td>$y = \frac{4}{x}$</td>
<td>Inverse</td>
</tr>
</tbody>
</table>

### Example 2: Writing an Inverse Variation Equation

The variables $x$ and $y$ vary inversely, and $y = 8$ when $x = 3$.

a. Write an equation that relates $x$ and $y$.

b. Find $y$ when $x = -4$.

**Solution**

a. Use the given values of $x$ and $y$ to find the constant of variation.

$$y = \frac{k}{x}$$

Write general equation for inverse variation.

$$8 = \frac{k}{3}$$

Substitute 8 for $y$ and 3 for $x$.

$$24 = k$$

Solve for $k$.

The inverse variation equation is $y = \frac{24}{x}$.

b. When $x = -4$, the value of $y$ is:

$$y = \frac{24}{-4} = -6$$
**EXAMPLE 3**  **Writing an Inverse Variation Model**

The speed of the current in a whirlpool varies inversely with the distance from the whirlpool’s center. The Lofoten Maelstrom is a whirlpool located off the coast of Norway. At a distance of 3 kilometers (3000 meters) from the center, the speed of the current is about 0.1 meter per second. Describe the change in the speed of the current as you move closer to the whirlpool’s center.

**SOLUTION**

First write an inverse variation model relating distance from center \(d\) and speed \(s\).

\[
s = \frac{k}{d} \quad \text{Model for inverse variation}
\]

\[
0.1 = \frac{k}{3000} \quad \text{Substitute 0.1 for } s \text{ and 3000 for } d.
\]

\[
300 = k \quad \text{Solve for } k.
\]

The model is \(s = \frac{300}{d}\). The table shows some speeds for different values of \(d\).

<table>
<thead>
<tr>
<th>Distance from center (meters), (d)</th>
<th>2000</th>
<th>1500</th>
<th>500</th>
<th>250</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed (meters per second), (s)</td>
<td>0.15</td>
<td>0.2</td>
<td>0.6</td>
<td>1.2</td>
<td>6</td>
</tr>
</tbody>
</table>

From the table you can see that the speed of the current increases as you move closer to the whirlpool’s center.

The equation for inverse variation can be rewritten as \(xy = k\). This tells you that a set of data pairs \((x, y)\) shows inverse variation if the products \(xy\) are constant or approximately constant.

**EXAMPLE 4**  **Checking Data for Inverse Variation**

**BIOLOGY CONNECTION** The table compares the wing flapping rate \(r\) (in beats per second) to the wing length \(l\) (in centimeters) for several birds. Do these data show inverse variation? If so, find a model for the relationship between \(r\) and \(l\).

<table>
<thead>
<tr>
<th>Bird</th>
<th>(r) (beats per second)</th>
<th>(l) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carrion crow</td>
<td>3.6</td>
<td>32.5</td>
</tr>
<tr>
<td>Common scoter</td>
<td>5.0</td>
<td>23.5</td>
</tr>
<tr>
<td>Great crested grebe</td>
<td>6.3</td>
<td>18.7</td>
</tr>
<tr>
<td>Curlew</td>
<td>4.0</td>
<td>29.2</td>
</tr>
<tr>
<td>Lesser black-backed gull</td>
<td>2.8</td>
<td>42.2</td>
</tr>
</tbody>
</table>

\(\text{Source: Smithsonian Miscellaneous Collections}\)

**SOLUTION**

Each product \(rl\) is approximately equal to 117. For instance, \((3.6)(32.5) = 117\) and \((5.0)(23.5) = 117.5\). So, the data do show inverse variation. A model for the relationship between wing flapping rate and wing length is \(r = \frac{117}{l}\).
GOAL 2 USING JOINT VARIATION

Joint variation occurs when a quantity varies directly as the product of two or more other quantities. For instance, if \( z = kxy \) where \( k \neq 0 \), then \( z \) varies jointly with \( x \) and \( y \). Other types of variation are also possible, as illustrated in the following example.

EXAMPLE 5 Comparing Different Types of Variation

Write an equation for the given relationship.

<table>
<thead>
<tr>
<th>RELATIONSHIP</th>
<th>EQUATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( y ) varies directly with ( x ).</td>
<td>( y = kx )</td>
</tr>
<tr>
<td>b. ( y ) varies inversely with ( x ).</td>
<td>( y = \frac{k}{x} )</td>
</tr>
<tr>
<td>c. ( z ) varies jointly with ( x ) and ( y ).</td>
<td>( z = kxy )</td>
</tr>
<tr>
<td>d. ( y ) varies inversely with the square of ( x ).</td>
<td>( y = \frac{k}{x^2} )</td>
</tr>
<tr>
<td>e. ( z ) varies directly with ( y ) and inversely with ( x ).</td>
<td>( z = \frac{ky}{x} )</td>
</tr>
</tbody>
</table>

EXAMPLE 6 Writing a Variation Model

SCIENCE CONNECTION The law of universal gravitation states that the gravitational force \( F \) (in newtons) between two objects varies jointly with their masses \( m_1 \) and \( m_2 \) (in kilograms) and inversely with the square of the distance \( d \) (in meters) between the two objects. The constant of variation is denoted by \( G \) and is called the universal gravitational constant.

a. Write an equation for the law of universal gravitation.

b. Estimate the universal gravitational constant.

Use the Earth and sun facts given at the right.

SOLUTION

a. \( F = \frac{Gm_1m_2}{d^2} \)

b. Substitute the given values and solve for \( G \).

\[
3.53 \times 10^{22} = \frac{G(5.98 \times 10^{24})(1.99 \times 10^{30})}{(1.50 \times 10^{11})^2} \\
3.53 \times 10^{22} = G(5.29 \times 10^{32}) \\
6.67 \times 10^{-11} = G
\]

The universal gravitational constant is about \( 6.67 \times 10^{-11} \) N \( \cdot \) m\(^2\)/kg\(^2\).

Mass of Earth: \( m_1 = 5.98 \times 10^{24} \) kg

Mass of sun: \( m_2 = 1.99 \times 10^{30} \) kg

Mean distance between Earth and sun: \( d = 1.50 \times 10^{11} \) m

Force between Earth and sun: \( F = 3.53 \times 10^{22} \) N
1. Complete this statement: If \( w \) varies directly as the product of \( x, y, \) and \( z \), then \( w \) varies \( \_ \_ \_ \) with \( x, y, \) and \( z \).

2. How can you tell whether a set of data pairs \((x, y)\) shows inverse variation?

3. Suppose \( z \) varies jointly with \( x \) and \( y \). What can you say about \( \frac{x}{z} \)?

4. \( \frac{x}{y} = \frac{1}{4} \)

5. \( \frac{x}{y} = 5 \)

6. \( y = x - 3 \)

7. \( x = \frac{7}{y} \)

8. \( \frac{y}{x} = 12 \)

9. \( \frac{1}{2}xy = 9 \)

10. \( y = \frac{1}{x} \)

11. \( 2x + y = 4 \)

Tell whether \( x \) and \( y \) show direct variation, inverse variation, or neither.

12. \( x = 15yz \)

13. \( \frac{x}{z} = 0.5y \)

14. \( xy = 4z \)

15. \( x = \frac{yz}{2} \)

16. \( x = \frac{3z}{y} \)

17. \( 2yz = 7x \)

18. \( \frac{x}{y} = 17z \)

19. \( 5x = 4yz \)

20. **Tools** The force \( F \) needed to loosen a bolt with a wrench varies inversely with the length \( l \) of the handle. Write an equation relating \( F \) and \( l \), given that 250 pounds of force must be exerted to loosen a bolt when using a wrench with a handle 6 inches long. How much force must be exerted when using a wrench with a handle 24 inches long?

### Practice and Applications

**Determining Variation** Tell whether \( x \) and \( y \) show direct variation, inverse variation, or neither.

21. \( xy = 10 \)

22. \( xy = \frac{1}{10} \)

23. \( y = x - 1 \)

24. \( \frac{y}{9} = x \)

25. \( x = \frac{5}{y} \)

26. \( 3x = y \)

27. \( x = 5y \)

28. \( x + y = 2.5 \)

**Inverse Variation Models** The variables \( x \) and \( y \) vary inversely. Use the given values to write an equation relating \( x \) and \( y \). Then find \( y \) when \( x = 2 \).

29. \( x = 5, \ y = -2 \)

30. \( x = 4, \ y = 8 \)

31. \( x = 7, \ y = 1 \)

32. \( x = \frac{1}{2}, \ y = 10 \)

33. \( x = -\frac{2}{3}, \ y = 6 \)

34. \( x = \frac{3}{4}, \ y = \frac{3}{8} \)

**Interpreting Data** Determine whether \( x \) and \( y \) show direct variation, inverse variation, or neither.

35. \[
\begin{array}{cc}
1.5 & 20 \\
2.5 & 12 \\
4 & 7.5 \\
5 & 6 \\
\end{array}
\]

36. \[
\begin{array}{cc}
31 & 217 \\
20 & 140 \\
17 & 119 \\
12 & 84 \\
\end{array}
\]

37. \[
\begin{array}{cc}
3 & 36 \\
7 & 105 \\
5 & 50 \\
16 & 48 \\
\end{array}
\]

38. \[
\begin{array}{cc}
4 & 16 \\
5 & 12.8 \\
1.6 & 40 \\
20 & 3.2 \\
\end{array}
\]
**JOINT VARIATION MODELS** The variable $z$ varies jointly with $x$ and $y$. Use the given values to write an equation relating $x$, $y$, and $z$. Then find $z$ when $x = -4$ and $y = 7$.

39. $x = 3, y = 8, z = 6$
40. $x = -12, y = 4, z = 2$
41. $x = 1, y = \frac{1}{3}, z = 5$
42. $x = -6, y = 3, z = \frac{2}{5}$
43. $x = \frac{5}{6}, y = \frac{3}{10}, z = 8$
44. $x = \frac{3}{8}, y = \frac{16}{17}, z = \frac{3}{2}$

**WRITING EQUATIONS** Write an equation for the given relationship.

45. $x$ varies inversely with $y$ and directly with $z$.
46. $y$ varies jointly with $z$ and the square root of $x$.
47. $w$ varies inversely with $x$ and jointly with $y$ and $z$.

**HOME REPAIR** In Exercises 48–50, use the following information.

On some tubes of caulk, the diameter of the circular nozzle opening can be adjusted to produce lines of varying thickness. The table shows the length $l$ of caulk obtained from a tube when the nozzle opening has diameter $d$ and cross-sectional area $A$.

48. Determine whether $l$ varies inversely with $d$. If so, write an equation relating $l$ and $d$.
49. Determine whether $l$ varies inversely with $A$. If so, write an equation relating $l$ and $A$.
50. Find the length of caulk you get from a tube whose nozzle opening has a diameter of $\frac{3}{4}$ inch.

**ASTRONOMY** In Exercises 51–53, use the following information.

A star’s diameter $D$ (as a multiple of the sun’s diameter) varies directly with the square root of the star’s luminosity $L$ (as a multiple of the sun’s luminosity) and inversely with the square of the star’s temperature $T$ (in kelvins).

51. Write an equation relating $D, L, T$, and a constant $k$.
52. The luminosity of Polaris is 10,000 times the luminosity of the sun. The surface temperature of Polaris is about 5800 kelvins. Using $k = 33,640,000$, find how the diameter of Polaris compares with the diameter of the sun.
53. The sun’s diameter is 1,390,000 kilometers. What is the diameter of Polaris?

54. **INTENSITY OF SOUND** The intensity $I$ of a sound (in watts per square meter) varies inversely with the square of the distance $d$ (in meters) from the sound’s source. At a distance of 1 meter from the stage, the intensity of the sound at a rock concert is about 10 watts per square meter. Write an equation relating $I$ and $d$. If you are sitting 15 meters back from the stage, what is the intensity of the sound you hear?

55. **SCIENCE CONNECTION** The work $W$ (in joules) done when lifting an object varies jointly with the mass $m$ (in kilograms) of the object and the height $h$ (in meters) that the object is lifted. The work done when a 120 kilogram object is lifted 1.8 meters is 2116.8 joules. Write an equation that relates $W, m,$ and $h$. How much work is done when lifting a 100 kilogram object 1.5 meters?
HEAT LOSS In Exercises 56 and 57, use the following information.
The heat loss \( h \) (in watts) through a single-pane glass window varies jointly with the window’s area \( A \) (in square meters) and the difference between the inside and outside temperatures \( d \) (in kelvins).

56. Write an equation relating \( h \), \( A \), \( d \), and a constant \( k \).

57. A single-pane window with an area of 1 square meter and a temperature difference of 1 kelvin has a heat loss of 5.7 watts. What is the heat loss through a single-pane window with an area of 2.5 square meters and a temperature difference of 20 kelvins?

58. GEOMETRY CONNECTION The area of a trapezoid varies jointly with the height and the sum of the lengths of the bases. When the sum of the lengths of the bases is 18 inches and the height is 4 inches, the area is 36 square inches. Find a formula for the area of a trapezoid.

59. MULTI-STEP PROBLEM The load \( P \) (in pounds) that can be safely supported by a horizontal beam varies jointly with the width \( W \) (in feet) of the beam and the square of its depth \( D \) (in feet), and inversely with its length \( L \) (in feet).
   
a. How does \( P \) change when the width and length of the beam are doubled?
   
b. How does \( P \) change when the width and depth of the beam are doubled?
   
c. How does \( P \) change when all three dimensions are doubled?
   
d. Writing Describe several ways a beam can be modified if the safe load it is required to support is increased by a factor of 4.

60. LOGICAL REASONING Suppose \( x \) varies inversely with \( y \) and \( y \) varies inversely with \( z \). How does \( x \) vary with \( z \)? Justify your answer algebraically.

MIXED REVIEW

SQUARE ROOT FUNCTIONS Graph the function. Then state the domain and range. (Review 7.5 for 9.2)

61. \( y = \sqrt{x} + 2 \)  
62. \( y = \sqrt{x} - 4 \)  
63. \( y = \sqrt{x} + 1 - 3 \)

SOLVING RADICAL EQUATIONS Solve the equation. Check for extraneous solutions. (Review 7.6)

64. \( \sqrt{x} = 22 \)  
65. \( \sqrt[4]{2x} + 2 = 6 \)  
66. \( x^{1/3} - 7 = 0 \)

67. \( \sqrt{x + 12} = 5 \)  
68. \( (x - 2)^{3/2} = -8 \)  
69. \( \sqrt{3x + 1} = \sqrt{x + 15} \)

70. COLLEGE ADMISSION The number of admission applications received by a college was 1152 in 1990 and increased 5% per year until 1998. (Review 8.1 for 9.2)
   
a. Write a model giving the number \( A \) of applications \( t \) years after 1990.
   
b. Graph the model. Use the graph to estimate the year in which there were 1400 applications.
Graphing Simple Rational Functions

**GOAL 1** Graphing a Simple Rational Function

A rational function is a function of the form

\[ f(x) = \frac{p(x)}{q(x)} \]

where \( p(x) \) and \( q(x) \) are polynomials and \( q(x) \neq 0 \). In this lesson you will learn to graph rational functions for which \( p(x) \) and \( q(x) \) are linear. For instance, consider the following rational function:

\[ y = \frac{1}{x} \]

The graph of this function is called a hyperbola and is shown below. Notice the following properties.

- The \( x \)-axis is a horizontal asymptote.
- The \( y \)-axis is a vertical asymptote.
- The domain and range are all nonzero real numbers.
- The graph has two symmetrical parts called branches. For each point \((x, y)\) on one branch, there is a corresponding point \((-x, -y)\) on the other branch.

### Investigating Graphs of Rational Functions

1. Graph each function.
   - a. \( y = \frac{2}{x} \)
   - b. \( y = \frac{3}{x} \)
   - c. \( y = \frac{-1}{x} \)
   - d. \( y = \frac{-2}{x} \)

2. Use the graphs to describe how the sign of \( a \) affects the graph of \( y = \frac{a}{x} \).

3. Use the graphs to describe how \( |a| \) affects the graph of \( y = \frac{a}{x} \).
All rational functions of the form \( y = \frac{a}{x-h} + k \) have graphs that are hyperbolas with asymptotes at \( x = h \) and \( y = k \). To draw the graph, plot a couple of points on each side of the vertical asymptote. Then draw the two branches of the hyperbola that approach the asymptotes and pass through the plotted points.

**EXAMPE 1 Graphing a Rational Function**

Graph \( y = \frac{-2}{x+3} - 1 \). State the domain and range.

**SOLUTION**

*Draw* the asymptotes \( x = -3 \) and \( y = -1 \).

*Plot* two points to the left of the vertical asymptote, such as \((-4, 1)\) and \((-5, 0)\), and two points to the right, such as \((-1, -2)\) and \((0, -\frac{5}{3})\).

*Use* the asymptotes and plotted points to draw the branches of the hyperbola.

The domain is all real numbers except \(-3\), and the range is all real numbers except \(-1\).

All rational functions of the form \( y = \frac{ax + b}{cx + d} \) also have graphs that are hyperbolas. The vertical asymptote occurs at the \( x \)-value that makes the denominator zero. The horizontal asymptote is the line \( y = \frac{a}{c} \).

**EXAMPLE 2 Graphing a Rational Function**

Graph \( y = \frac{x + 1}{2x - 4} \). State the domain and range.

**SOLUTION**

*Draw* the asymptotes. Solve \( 2x - 4 = 0 \) for \( x \) to find the vertical asymptote \( x = 2 \). The horizontal asymptote is the line \( y = \frac{a}{c} = \frac{1}{2} \).

*Plot* two points to the left of the vertical asymptote, such as \( \left(0, -\frac{1}{4}\right) \) and \( (1, -1) \), and two points to the right, such as \( (3, 2) \) and \( \left(4, \frac{5}{4}\right) \).

*Use* the asymptotes and plotted points to draw the branches of the hyperbola.

The domain is all real numbers except \( 2 \), and the range is all real numbers except \( \frac{1}{2} \).
**USING RATIONAL FUNCTIONS IN REAL LIFE**

**EXAMPLE 3 Writing a Rational Model**

For a fundraising project, your math club is publishing a fractal art calendar. The cost of the digital images and the permission to use them is $850. In addition to these “one-time” charges, the unit cost of printing each calendar is $3.25.

**a.** Write a model that gives the average cost per calendar as a function of the number of calendars printed.

**b.** Graph the model and use the graph to estimate the number of calendars you need to print before the average cost drops to $5 per calendar.

**c.** Describe what happens to the average cost as the number of calendars printed increases.

**SOLUTION**

**a.** The average cost is the total cost of making the calendars divided by the number of calendars printed.

\[
\text{Average cost} = \frac{\text{One-time charges} + \text{Unit cost} \cdot \text{Number printed}}{\text{Number printed}}
\]

**b.** The graph of the model is shown at the right. The \(A\)-axis is the vertical asymptote and the line \(A = 3.25\) is the horizontal asymptote. The domain is \(x > 0\) and the range is \(A > 3.25\). When \(A = 5\) the value of \(x\) is about 500. So, you need to print about 500 calendars before the average cost drops to $5 per calendar.

**c.** As the number of calendars printed increases, the average cost per calendar gets closer and closer to $3.25. For instance, when \(x = 5000\) the average cost is $3.42, and when \(x = 10,000\) the average cost is $3.34.
1. Complete this statement: The graph of a function of the form \( y = \frac{a}{x-h} + k \) is called a(n) \( \underline{?} \).

2. **ERROR ANALYSIS** Explain why the graph shown is not the graph of \( y = \frac{6}{x+3} + 7 \).

3. If the graph of a rational function is a hyperbola with the \( x \)-axis and the \( y \)-axis as asymptotes, what is the domain of the function? What is the range?

4. Identify the horizontal and vertical asymptotes of the graph of the function.
   
   \( y = \frac{2}{x-3} + 4 \)
   
   \( y = \frac{2x + 3}{x + 4} \)
   
   \( y = \frac{x - 3}{x + 3} \)

5. \( y = \frac{2x + 3}{x + 4} \)

6. \( y = \frac{x - 3}{x + 3} \)

7. \( y = \frac{x + 5}{2x - 4} \)

8. \( y = \frac{3}{x + 8} - 10 \)

9. \( y = \frac{-4}{x - 6} - 5 \)

10. **CALENDAR FUNDRAISER** Look back at Example 3 on page 542. Suppose you decide to generate your own fractals on a computer to save money. The cost for the software (a “one-time” cost) is $125. Write a model that gives the average cost per calendar as a function of the number of calendars printed. Graph the model and use the graph to estimate the number of calendars you need to print before the average cost drops to $5 per calendar.

**PRACTICE AND APPLICATIONS**

**IDENTIFYING ASYMPTOTES** Identify the horizontal and vertical asymptotes of the graph of the function. Then state the domain and range.

11. \( y = \frac{3}{x} + 2 \)

12. \( y = \frac{4}{x - 3} + 2 \)

13. \( y = \frac{-2}{x + 3} - 2 \)

14. \( y = \frac{x + 2}{x - 3} \)

15. \( y = \frac{2x + 2}{3x + 1} \)

16. \( y = \frac{-3x + 2}{-4x - 5} \)

17. \( y = \frac{-22}{x + 43} - 17 \)

18. \( y = \frac{34x - 2}{16x + 4} \)

19. \( y = \frac{4}{x - 6} + 19 \)

**MATCHING GRAPHS** Match the function with its graph.

20. \( y = \frac{3}{x - 2} + 3 \)

21. \( y = \frac{-3}{x - 2} + 3 \)

22. \( y = \frac{x + 2}{x + 3} \)
GRAPHING FUNCTIONS Graph the function. State the domain and range.

23. \( y = \frac{4}{x} \)

24. \( y = \frac{3}{x - 3} + 1 \)

25. \( y = \frac{-4}{x + 5} - 8 \)

26. \( y = \frac{1}{x - 7} + 3 \)

27. \( y = \frac{6}{x + 2} - 6 \)

28. \( y = \frac{5}{x} + 4 \)

29. \( y = \frac{1}{4x + 12} - 2 \)

30. \( y = \frac{3}{2x} \)

31. \( y = \frac{4}{3x - 6} + 5 \)

GRAPHING FUNCTIONS Graph the function. State the domain and range.

32. \( y = \frac{x + 2}{x + 3} \)

33. \( y = \frac{x}{4x + 3} \)

34. \( y = \frac{x - 7}{3x - 8} \)

35. \( y = \frac{9x + 1}{3x - 2} \)

36. \( y = \frac{-3x + 10}{4x - 12} \)

37. \( y = \frac{5x + 2}{4x} \)

38. \( y = \frac{3x}{2x - 4} \)

39. \( y = \frac{7x}{-x - 15} \)

40. \( y = \frac{-14x - 4}{2x - 1} \)

41. CRITICAL THINKING Write a rational function that has the vertical asymptote \( x = -4 \) and the horizontal asymptote \( y = 3 \).

RACQUETBALL In Exercises 42 and 43, use the following information.
You’ve paid $120 for a membership to a racquetball club. Court time is $5 per hour.

42. Write a model that represents your average cost per hour of court time as a function of the number of hours played. Graph the model. What is an equation of the horizontal asymptote and what does the asymptote represent?

43. Suppose that you can play racquetball at the YMCA for $9 per hour without being a member. How many hours would you have to play at the racquetball club before your average cost per hour of court time is less than $9?

44. LIGHTNING Air temperature affects how long it takes sound to travel a given distance. The time it takes for sound to travel one kilometer can be modeled by

\[ t = \frac{1000}{0.6T + 331} \]

where \( t \) is the time (in seconds) and \( T \) is the temperature (in degrees Celsius). You are 1 kilometer from a lightning strike and it takes you exactly 3 seconds to hear the sound of thunder. Use a graph to find the approximate air temperature. (Hint: Use tick marks that are 0.1 unit apart on the \( t \)-axis.)

ECONOMICS In Exercises 45 and 46, use the following information.
Economist Arthur Laffer argues that beyond a certain percent \( p_m \), increased taxes will produce less government revenue. His theory is illustrated in the graph below.

45. Using Laffer’s theory, an economist models the revenue generated by one kind of tax by

\[ R = \frac{80p - 8000}{p - 110} \]

where \( R \) is the government revenue (in tens of millions of dollars) and \( p \) is the percent tax rate (35 \( \leq p \leq 100 \)). Graph the model.

46. Use your graph from Exercise 45 to find the tax rate that yields $600 million of revenue.
**Doppler Effect** In Exercises 47 and 48, use the following information.

When the source of a sound is moving relative to a stationary listener, the frequency \( f_l \) (in hertz) heard by the listener is different from the frequency \( f_s \) (in hertz) of the sound at its source. An equation for the frequency heard by the listener is

\[
f_l = \frac{740f_s}{740 - r}
\]

where \( r \) is the speed (in miles per hour) of the sound source relative to the listener.

47. The sound of an ambulance siren has a frequency of about 2000 hertz. You are standing on the sidewalk as an ambulance approaches with its siren on. Write the frequency that you hear as a function of the ambulance’s speed.

48. Graph the function from Exercise 47 for \( 0 \leq r \leq 60 \). What happens to the frequency you hear as the value of \( r \) increases?

49. Writing In what line(s) is the graph of \( y = \frac{x}{x^2 + 1} \) symmetric? What does this symmetry tell you about the inverse of the function \( f(x) = \frac{1}{x} + 1 \)?

50. **Multiple Choice** What are the asymptotes of the graph of \( y = \frac{-73}{x - 141} + 27 \)?

A) \( x = 141, y = 27 \)  
B) \( x = -141, y = 27 \)  
C) \( x = -73, y = 27 \)  
D) \( x = -73, y = 141 \)  
E) None of these

51. **Multiple Choice** Which of the following is a function whose domain and range are all nonzero real numbers?

A) \( f(x) = \frac{x}{2x + 1} \)  
B) \( f(x) = \frac{2x - 1}{3x - 2} \)  
C) \( f(x) = \frac{1}{x} + 1 \)  
D) \( f(x) = \frac{x - 2}{x} \)  
E) None of these

52. **Equivalent Forms** Show algebraically that the function \( f(x) = \frac{3}{x - 5} + 10 \) and the function \( g(x) = \frac{10x - 47}{x - 5} \) are equivalent.

---

**Mixed Review**

**Graphing Polynomials** Graph the polynomial function. (Review 6.2 for 9.3)

53. \( f(x) = 3x^5 \)  
54. \( f(x) = 4 - 2x^3 \)  
55. \( f(x) = x^6 - 1 \)

56. \( f(x) = 4x^4 + 1 \)  
57. \( f(x) = 6x^7 \)  
58. \( f(x) = x^3 - 5 \)

**Factoring** Factor the polynomial. (Review 6.4 for 9.3)

59. \( 8x^3 - 125 \)  
60. \( 3x^3 + 81 \)  
61. \( x^3 + 3x^2 + 3x + 9 \)

62. \( 5x^3 + 10x^2 + x + 2 \)  
63. \( 81x^4 - 1 \)  
64. \( 4x^4 - 4x^2 - 120 \)

**Simplifying Expressions** Simplify the expression. (Review 8.3)

65. \( e^x \)  
66. \( 7e^{-5}e^8 \)  
67. \( e^3e^{4x} + 1 \)

68. \( 6e^x \)  
69. \( e^4e^{-2}e^{-3x} \)  
70. \( e^3e^{-5} \)

---

9.2 Graphing Simple Rational Functions
Graphing Rational Functions

You can use a graphing calculator to graph rational functions.

**EXAMPLE**

Use a graphing calculator to graph \( y = \frac{x + 2}{x - 2} \).

**SOLUTION**

Begin by entering the function, using parentheses as shown.

Most graphing calculators have two graphing modes: Connected mode and Dot mode. The graphs below show the function graphed in each mode.

Notice that the graph on the left has a vertical line at approximately \( x = 2 \). This line is not part of the graph—it is simply the graphing calculator’s attempt at connecting the two branches of the graph.

Be sure to choose a viewing window that shows all of the important characteristics of the graph. For instance, in the graph shown at the right, the viewing window is inadequate because it does not show the two branches of the graph of the rational function.

**EXERCISES**

Use a graphing calculator to graph the rational function. Choose a viewing window that displays the important characteristics of the graph.

1. \( y = \frac{7}{x} + 3 \)
2. \( y = 9 - \frac{5}{x} \)
3. \( y = 5 + \frac{3}{x - 6} \)
4. \( y = \frac{4}{x} + 6 \)
5. \( y = \frac{x - 5}{x + 1} \)
6. \( y = \frac{8 - 5x}{x - 5} \)

7. **DELIVERY CHARGES** You and your friends order pizza and have it delivered to your house. The restaurant charges $8 per pizza plus a $2 delivery fee. Write a model that gives the average cost per pizza as a function of the number of pizzas ordered. Graph the model. Describe what happens to the average cost as the number of pizzas ordered increases.
Graphing General Rational Functions

GOAL 1  GRAPHING RATIONAL FUNCTIONS

In Lesson 9.2 you learned how to graph rational functions of the form

\[ f(x) = \frac{p(x)}{q(x)} \]

for which \( p(x) \) and \( q(x) \) are linear polynomials and \( q(x) \neq 0 \). In this lesson you will learn how to graph rational functions for which \( p(x) \) and \( q(x) \) may be higher-degree polynomials.

Graphing a Rational Function \((m < n)\)

Graph \( y = \frac{4}{x^2 + 1} \). State the domain and range.

SOLUTION

The numerator has no zeros, so there is no \( x \)-intercept. The denominator has no real zeros, so there is no vertical asymptote. The degree of the numerator (0) is less than the degree of the denominator (2), so the line \( y = 0 \) (the \( x \)-axis) is a horizontal asymptote. The bell-shaped graph passes through the points \((-3, 0.4), (-1, 2), (0, 4), (1, 2), \) and \((3, 0.4)\). The domain is all real numbers, and the range is \( 0 < y \leq 4 \).
EXAMPLE 2  **Graphing a Rational Function (m = n)**

Graph \( y = \frac{3x^2}{x^2 - 4} \)

**SOLUTION**

The numerator has 0 as its only zero, so the graph has one \( x \)-intercept at \( (0, 0) \). The denominator can be factored as \((x + 2)(x - 2)\), so the denominator has zeros \(-2\) and 2. This implies that the lines \( x = -2 \) and \( x = 2 \) are vertical asymptotes of the graph. The degree of the numerator (2) is equal to the degree of the denominator (2), so the horizontal asymptote is \( y = \frac{a_m}{b_n} = 3 \). To draw the graph, plot points between and beyond the vertical asymptotes.

\[
\begin{array}{c|c}
 x & y \\
-4 & 4 \\
-3 & 5.4 \\
-1 & -1 \\
0 & 0 \\
1 & -1 \\
3 & 5.4 \\
4 & 4 \\
\end{array}
\]

EXAMPLE 3  **Graphing a Rational Function (m > n)**

Graph \( y = \frac{x^2 - 2x - 3}{x + 4} \).

**SOLUTION**

The numerator can be factored as \((x - 3)(x + 1)\), so the \( x \)-intercepts of the graph are 3 and \(-1\). The only zero of the denominator is \(-4\), so the only vertical asymptote is \( x = -4 \). The degree of the numerator (2) is greater than the degree of the denominator (1), so there is no horizontal asymptote and the end behavior of the graph of \( f \) is the same as the end behavior of the graph of \( y = x^2 - 1 = x \). To draw the graph, plot points to the left and right of the vertical asymptote.

\[
\begin{array}{c|c}
 x & y \\
-12 & -20.6 \\
-9 & -19.2 \\
-6 & -22.5 \\
-2 & 2.5 \\
0 & -0.75 \\
2 & -0.5 \\
4 & 0.63 \\
6 & 2.1 \\
\end{array}
\]
GOAL 2 USING RATIONAL FUNCTIONS IN REAL LIFE

Manufacturers often want to package their products in a way that uses the least amount of packaging material. Finding the most efficient packaging sometimes involves finding a local minimum of a rational function.

EXAMPLE 4 Finding a Local Minimum

A standard beverage can has a volume of 355 cubic centimeters.

a. Find the dimensions of the can that has this volume and uses the least amount of material possible.

b. Compare your result with the dimensions of an actual beverage can, which has a radius of 3.1 centimeters and a height of 11.8 centimeters.

SOLUTION

a. The volume must be 355 cubic centimeters, so you can write the height $h$ of each possible can in terms of its radius $r$.

$$V = \pi r^2 h \quad \text{Formula for volume of cylinder}$$

$$355 = \pi r^2 h \quad \text{Substitute 355 for } V.$$ 

$$\frac{355}{\pi r^2} = h \quad \text{Solve for } h.$$ 

Using the least amount of material is equivalent to having a minimum surface area $S$. You can find the minimum surface area by writing its formula in terms of a single variable and graphing the result.

$$S = 2\pi r^2 + 2\pi rh \quad \text{Formula for surface area of cylinder}$$

$$= 2\pi r^2 + 2\pi r \left( \frac{355}{\pi r^2} \right) \quad \text{Substitute for } h.$$ 

$$= 2\pi r^2 + \frac{710}{r} \quad \text{Simplify.}$$

Graph the function for the surface area $S$ using a graphing calculator. Then use the Minimum feature to find the minimum value of $S$. When you do this, you get a minimum value of about 278, which occurs when $r \approx 3.84$ centimeters and

$$h \approx \frac{355}{\pi (3.84)^2} \approx 7.66 \text{ centimeters.}$$

b. An actual beverage can is taller and narrower than the can with minimal surface area—probably to make it easier to hold the can in one hand.
1. Let \( f(x) = \frac{p(x)}{q(x)} \) where \( p(x) \) and \( q(x) \) are polynomials with no common factors other than 1. Complete this statement: The line \( y = 0 \) is a horizontal asymptote of the graph of \( f \) when the degree of \( q(x) \) is \( \frac{?}{\text{degree of } p(x)} \).

2. Let \( f(x) = \frac{p(x)}{q(x)} \) where \( p(x) \) and \( q(x) \) are polynomials with no common factors other than 1. Describe how to find the \( x \)-intercepts and the vertical asymptotes of the graph of \( f \).

3. Let \( f(x) = \frac{p(x)}{q(x)} \) where \( p(x) \) and \( q(x) \) are both cubic polynomials with no common factors other than 1. The leading coefficient of \( p(x) \) is 8 and the leading coefficient of \( q(x) \) is 2. Describe the end behavior of the graph of \( f \).

10. **SOUP CANS** The can for a popular brand of soup has a volume of about 342 cubic centimeters. Find the dimensions of the can with this volume that uses the least metal possible. Compare these dimensions with the dimensions of the actual can, which has a radius of 3.3 centimeters and a height of 10 centimeters.

11. Identify the \( x \)-intercepts and vertical asymptotes of the graph of the function.

12. \( y = \frac{x^2 + 3}{x - 1} \)

13. \( y = \frac{2x^2 - 9x - 5}{x^2 - 16} \)

14. \( y = \frac{2x + 3}{x^3} \)

15. \( y = \frac{x^2 + 4x - 5}{x - 6} \)

16. \( y = \frac{3x^2 - 13x + 10}{x^2 + 8} \)

17. \( y = \frac{2x + 8}{3x^2 - 9} \)

18. \( y = \frac{2x^2 + x}{x^2 + 1} \)

19. \( y = \frac{x^3 - 27}{2x} \)

20. \( y = \frac{x^2 - 7}{x^2 + 2} \)

21. \( y = \frac{-8}{x^2 - 4} \)

22. \( y = \frac{x^3}{x^2 - 4} \)

**MATCHING GRAPHS** Match the function with its graph.

A. [Graph A]

B. [Graph B]

C. [Graph C]
MATCHING GRAPHS  Match the function with its graph.

23. \( y = \frac{3}{x^3 - 27} \)  
   A.  
   
24. \( y = -\frac{x^3}{x^2 + 9} \)  
   B.  
   
25. \( y = \frac{x^2 + 4x}{2x - 1} \)  
   C.  

GRAPHING FUNCTIONS  Graph the function.

26. \( y = \frac{2x^2 - 3}{x + 2} \)  
27. \( y = -\frac{24}{x^2 + 8} \)  
28. \( y = \frac{x^2 - 4}{x^2 + 3} \)  
29. \( y = \frac{4x + 1}{x^2 - 1} \)  
30. \( y = \frac{2x^2 + 3x + 1}{x^2 - 5x + 4} \)  
31. \( y = -\frac{2x^2}{3x + 6} \)  
32. \( y = \frac{3x^3 + 1}{4x^3 - 32} \)  
33. \( y = \frac{x^2 - 11x - 12}{x^3 + 27} \)  
34. \( y = \frac{4 - x}{5x^2 - 4x - 1} \)  
35. \( y = -\frac{4x^2}{x^2 - 16} \)  
36. \( y = \frac{x^2 - 9x + 20}{2x} \)  
37. \( y = \frac{x^3 + 5x^2 - 1}{x^2 - 4x} \)  

38. **GARDEN FENCING**  Suppose you want to make a rectangular garden with an area of 200 square feet. You want to use the side of your house for one side of the garden and use fencing for the other three sides. Find the dimensions of the garden that minimize the length of fencing needed.

39. **ENERGY EXPENDITURE**  The total energy expenditure \( E \) (in joules per gram mass per kilometer) of a typical budgerigar parakeet can be modeled by

\[
E = \frac{0.31v^2 - 21.7v + 471.75}{v}
\]

where \( v \) is the speed of the bird (in kilometers per hour). Graph the model. What speed minimizes a budgerigar’s energy expenditure?  
   ▶ Source: *Introduction to Mathematics for Life Scientists*

40. **OCEANOGRAPHY**  The mean temperature \( T \) (in degrees Celsius) of the Atlantic Ocean between latitudes 40°N and 40°S can be modeled by

\[
T = \frac{-17,800d + 20,000}{3d^2 + 740d + 1000}
\]

where \( d \) is the depth (in meters). Graph the model. Use your graph to estimate the depth at which the mean temperature is 4°C.  
   ▶ Source: *Practical Handbook of Marine Science*

41. **HOSPITAL COSTS**  For 1985 to 1995, the average daily cost per patient \( C \) (in dollars) at community hospitals in the United States can be modeled by

\[
C = \frac{-22,407x + 462,048}{5x^2 - 122x + 1000}
\]

where \( x \) is the number of years since 1985. Graph the model. Would you use this model to predict patient costs in 2005? Explain.  
   ▶ Source: *Hospital Statistics*
42. **Automotive Industry** For 1980 to 1995, the total revenue $R$ (in billions of dollars) from parking and automotive service and repair in the United States can be modeled by

$$R = \frac{427x^2 - 6416x + 30,432}{-0.7x^3 + 25x^2 - 268x + 1000}$$

where $x$ is the number of years since 1980. Graph the model. In what year was the total revenue approximately $75$ billion?

**Data Update** of U.S. Bureau of the Census data at www.mcdougallittell.com

**Science Connection** In Exercises 43–45, use the following information.

The acceleration due to gravity $g'$ (in meters per second squared) of a falling object at the moment it is dropped is given by the function

$$g' = \frac{3.99 \times 10^{14}}{h^2 + (1.28 \times 10^7)h + 4.07 \times 10^{13}}$$

where $h$ is the object’s altitude (in meters) above sea level.

43. Graph the function.

44. What is the acceleration due to gravity for an object dropped at an altitude of 5000 kilometers?

45. Describe what happens to $g'$ as $h$ increases.

46. **Critical Thinking** Give an example of a rational function whose graph has two vertical asymptotes: $x = 2$ and $x = 7$.

47. **Multiple Choice** What is the horizontal asymptote of the graph of the following function?

$$y = \frac{10x^2 - 1}{x^3 + 8}$$

- **A** $y = -10$
- **B** $y = 0$
- **C** $y = 2$
- **D** $y = 10$
- **E** No horizontal asymptote

48. **Multiple Choice** Which of the following functions is graphed?

- **A** $y = \frac{-5x^2}{x^2 + 9}$
- **B** $y = \frac{5x^2}{x^2 - 9}$
- **C** $y = \frac{5x^2}{x^2 + 9}$
- **D** $y = \frac{-5x^2}{x^2 - 9}$

49. Consider the following two functions:

$$f(x) = \frac{(x + 1)(x + 2)}{(x - 3)(x - 5)}$$

and

$$g(x) = \frac{(x + 2)(x - 3)}{(x - 3)(x - 5)}$$

Notice that the numerator and denominator of $g$ have a common factor of $x - 3$.

a. Make a table of values for each function from $x = 2.95$ to $x = 3.05$ in increments of 0.01.

b. Use your table of values to graph each function for $2.95 \leq x \leq 3.05$.

c. As $x$ approaches 3, what happens to the graph of $f(x)$? to the graph of $g(x)$?

d. What do you think is true about the graph of a function $g(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ have a common factor $x - k$?
**Mixed Review**

**Simplifying Algebraic Expressions** Simplify the expression. Tell which properties of exponents you used. (Review 6.1 for 9.4)

50. \( \frac{x^3y}{xy^4} \)
51. \( \frac{x^6y}{xy} \)
52. \( \frac{3x^3y^3}{6x^{-1}y} \)
53. \( \frac{12x^5y^2}{3x^{-2}y^5} \)
54. \( \left( \frac{x^2y}{x^3y} \right)^2 \)
55. \( \left( \frac{5x^3}{25xy^2} \right)^3 \)

**Joint Variation Models** The variable \( z \) varies jointly with \( x \) and \( y \). Use the given values to write an equation relating \( x, y, \) and \( z \). Then find \( z \) when \( x = -3 \) and \( y = 2 \). (Review 9.1)

56. \( x = 3, y = -6, z = 2 \)
57. \( x = -5, y = 2, z = \frac{3}{4} \)
58. \( x = -8, y = 4, z = \frac{8}{3} \)
59. \( x = 1, y = \frac{1}{2}, z = 4 \)

**Verifying Inverses** Verify that \( f \) and \( g \) are inverse functions. (Review 7.4)

60. \( f(x) = \frac{1}{2}x - 3, g(x) = 2x + 6 \)
61. \( f(x) = -3x + 2, g(x) = -\frac{1}{3}x + \frac{2}{3} \)
62. \( f(x) = 5x^3 + 2, g(x) = \left( \frac{x - 2}{5} \right)^{1/3} \)
63. \( f(x) = 16x^4, x \geq 0; g(x) = \frac{\sqrt[4]{x}}{2} \)

**Quiz 1**

The variables \( x \) and \( y \) vary inversely. Use the given values to write an equation relating \( x \) and \( y \). Then find \( y \) when \( x = -3 \). (Lesson 9.1)

1. \( x = 6, y = -2 \)
2. \( x = 11, y = 6 \)
3. \( x = \frac{1}{5}, y = 30 \)

The variable \( x \) varies jointly with \( y \) and \( z \). Use the given values to write an equation relating \( x, y, \) and \( z \). Then find \( y \) when \( x = 4 \) and \( z = 1 \). (Lesson 9.1)

4. \( x = 5, y = -5, z = 6 \)
5. \( x = 12, y = 6, z = \frac{1}{2} \)
6. \( x = -10, y = 2, z = 4 \)

Graph the function. (Lessons 9.2 and 9.3)

7. \( y = \frac{10}{x} \)
8. \( y = \frac{2}{x + 9} - 7 \)
9. \( y = \frac{3x + 5}{2x} \)
10. \( y = \frac{6x}{x^2 - 36} \)
11. \( y = \frac{3x^2}{x^2 - 25} \)
12. \( y = \frac{x^2 - 6x - 5}{x + 2} \)

13. **Hotel Revenue** For 1980 to 1995, the total revenue \( R \) (in billions of dollars) from hotels and motels in the United States can be modeled by

\[
R = \frac{2.76x + 26.88}{-0.01x + 1}
\]

where \( x \) is the number of years since 1980. Graph the model. Use your graph to find the year in which the total revenue from hotels and motels was approximately $68 billion. (Lesson 9.2)
Multiplying and Dividing Rational Expressions

**Goal 1** Working with Rational Expressions

A rational expression is in simplified form provided its numerator and denominator have no common factors (other than ±1). To simplify a rational expression, apply the following property.

\[
\frac{ac}{bd} = \frac{a}{b} \quad \text{Divide out common factor } c.
\]

Simplifying a rational expression usually requires two steps. First, factor the numerator and denominator. Then, divide out any factors that are common to both the numerator and denominator. Here is an example:

\[
\frac{x^2 + 5x}{x^2} = \frac{x(x + 5)}{x \cdot x} = \frac{x + 5}{x}
\]

Notice that you can divide out common factors in the second expression above, but you cannot divide out like terms in the third expression.

**Example 1** Simplifying a Rational Expression

Simplify: \(\frac{x^2 - 4x - 12}{x^2 - 4}\)

**Solution**

\[
\frac{x^2 - 4x - 12}{x^2 - 4} = \frac{(x + 2)(x - 6)}{(x + 2)(x - 2)} \quad \text{Factor numerator and denominator.}
\]

\[
= \frac{x - 6}{x - 2} \quad \text{Divide out common factor.}
\]

The rule for multiplying rational expressions is the same as the rule for multiplying numerical fractions: multiply numerators, multiply denominators, and write the new fraction in simplified form.

\[
\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \quad \text{Simplify } \frac{ac}{bd} \text{ if possible.}
\]
**EXAMPLE 2**  \(\text{Multiplying Rational Expressions Involving Monomials}\)

Multiply: \(\frac{5x^2y}{2xy^3} \cdot \frac{6x^3y^2}{10y}\)

**SOLUTION**

\[
\frac{5x^2y}{2xy^3} \cdot \frac{6x^3y^2}{10y} = \frac{30x^5y^3}{20xy^4} = \frac{3x^4}{2y}
\]

Multiply numerators and denominators.

Factor and divide out common factors.

Simplified form

**EXAMPLE 3**  \(\text{Multiplying Rational Expressions Involving Polynomials}\)

Multiply: \(\frac{4x - 4x^2}{x^2 + 2x - 3} \cdot \frac{x^2 + x - 6}{4x}\)

**SOLUTION**

\[
\frac{4x - 4x^2}{x^2 + 2x - 3} \cdot \frac{x^2 + x - 6}{4x} = \frac{4x(1 - x)}{(x - 1)(x + 3)} \cdot \frac{(x + 3)(x - 2)}{4x}
\]

Factor numerators and denominators.

Multiply numerators and denominators.

Rewrite \(1 - x\) as \((-1)(x - 1)\).

Divide out common factors.

Simplified form

**EXAMPLE 4**  \(\text{Multiplying by a Polynomial}\)

Multiply: \(\frac{x + 3}{8x^3 - 1} \cdot (4x^2 + 2x + 1)\)

**SOLUTION**

\[
\frac{x + 3}{8x^3 - 1} \cdot (4x^2 + 2x + 1) = \frac{x + 3}{(2x - 1)(4x^2 + 2x + 1)} \cdot \frac{4x^2 + 2x + 1}{2x - 1}
\]

Write polynomial as rational expression.

Factor and multiply numerators and denominators.

Divide out common factors.

Simplified form

---

**Look Back**

For help with factoring a difference of two cubes, see p. 345.
To divide one rational expression by another, multiply the first expression by the reciprocal of the second expression.

\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \quad \text{Simplify} \quad \frac{ad}{bc} \text{ if possible.}
\]

**EXAMPLE 5**  
**Dividing Rational Expressions**

Divide: \[
\frac{5x}{3x - 12} \div \frac{x^2 - 2x}{x^2 - 6x + 8}
\]

**SOLUTION**

\[
\frac{5x}{3x - 12} \div \frac{x^2 - 2x}{x^2 - 6x + 8} = \frac{5x}{3x - 12} \cdot \frac{x^2 - 6x + 8}{x^2 - 2x}
\]

\[
= \frac{5x}{3(x - 4)} \cdot \frac{(x - 2)(x - 4)}{x(x - 2)}
\]

\[
= \frac{5x(x - 2)(x - 4)}{3(x - 4)(x - 2)}
\]

\[
= \frac{5}{3}
\]

**EXAMPLE 6**  
**Dividing by a Polynomial**

Divide: \[
\frac{6x^2 + 7x - 3}{6x^2} \div (2x^2 + 3x)
\]

**SOLUTION**

\[
\frac{6x^2 + 7x - 3}{6x^2} \div (2x^2 + 3x) = \frac{6x^2 + 7x - 3}{6x^2} \cdot \frac{1}{2x^2 + 3x}
\]

\[
= \frac{(3x - 1)(2x + 3)}{6x^2} \cdot \frac{1}{x(2x + 3)}
\]

\[
= \frac{(3x - 1)(2x + 3)}{(6x^2)(x)(2x + 3)}
\]

\[
= \frac{3x - 1}{6x^3}
\]

**EXAMPLE 7**  
**Multiplying and Dividing**

Simplify: \[
\frac{x}{x + 5} \cdot (3x - 5) \div \frac{9x^2 - 25}{x + 5}
\]

**SOLUTION**

\[
\frac{x}{x + 5} \cdot (3x - 5) \div \frac{9x^2 - 25}{x + 5} = \frac{x}{x + 5} \cdot \frac{3x - 5}{1} \cdot \frac{x + 5}{9x^2 - 25}
\]

\[
= \frac{x(3x - 5)(x + 5)}{(x + 5)(3x - 5)(3x + 5)}
\]

\[
= \frac{x}{3x + 5}
\]
**EXAMPLE 8** *Writing and Simplifying a Rational Model*

**SKYDIVING** A falling skydiver accelerates until reaching a constant falling speed, called the *terminal velocity*. Because of air resistance, the ratio of a skydiver’s volume to his or her cross-sectional surface area affects the terminal velocity: the larger the ratio, the greater the terminal velocity.

**a.** The diagram shows a simplified geometric model of a skydiver with maximum cross-sectional surface area. Use the diagram to write a model for the ratio of volume to cross-sectional surface area for a skydiver.

**b.** Use the result of part (a) to compare the terminal velocities of two skydivers: one who is 60 inches tall and one who is 72 inches tall.

**Solution**

**a.** The volume and cross-sectional surface area of each part of the skydiver are given in the table below. (Assume that the front side of the skydiver’s body is parallel with the ground when falling.)

<table>
<thead>
<tr>
<th>Body part</th>
<th>Volume</th>
<th>Cross-sectional surface area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arm or leg</td>
<td>$V = 6x^3$</td>
<td>$S = 6x(x) = 6x^2$</td>
</tr>
<tr>
<td>Head</td>
<td>$V = 8x^3$</td>
<td>$S = 2x(2x) = 4x^2$</td>
</tr>
<tr>
<td>Trunk</td>
<td>$V = 72x^3$</td>
<td>$S = 6x(4x) = 24x^2$</td>
</tr>
</tbody>
</table>

Using these volumes and cross-sectional surface areas, you can write the ratio as:

$$\frac{\text{Volume}}{\text{Surface area}} = \frac{4(6x^3) + 8x^3 + 72x^3}{4(6x^2) + 4x^2 + 24x^2}$$

$$= \frac{104x^3}{52x^2}$$

$$= 2x$$

**b.** The overall height of the geometric model is $14x$. For the skydiver whose height is 60 inches, $14x = 60$, so $x = 4.3$. For the skydiver whose height is 72 inches, $14x = 72$, so $x = 5.1$. The ratio of volume to cross-sectional surface area for each skydiver is:

- **60 inch skydiver:** $\frac{\text{Volume}}{\text{Surface area}} = 2x = 2(4.3) = 8.6$
- **72 inch skydiver:** $\frac{\text{Volume}}{\text{Surface area}} = 2x = 2(5.1) = 10.2$

The taller skydiver has the greater terminal velocity.
**GUIDED PRACTICE**

**Vocabulary Check**  
1. Explain how you know when a rational expression is in simplified form.

**Concept Check**  
2. **ERROR ANALYSIS** Explain what is wrong with the simplification of the rational expression shown.

**Skill Check**  
If possible, simplify the rational expression.

3. \[ \frac{4x^2}{4x^3 + 12x} \]
4. \[ \frac{x^2 + 4x - 5}{x^2 - 1} \]
5. \[ \frac{x^2 + 10x - 4}{x^2 + 10x} \]
6. \[ \frac{6x^2 - 4x - 3}{3x^2 + x} \]
7. \[ \frac{x^2 - 9}{2x + 1} \]
8. \[ \frac{2x^3 - 32x}{x^2 + 8x + 16} \]

Perform the indicated operation. Simplify the result.

9. \[ \frac{16x^3}{5y^2} \cdot \frac{7x^4y^3}{80y^3} \]
10. \[ \frac{7x^2y^3}{5xy} \cdot \frac{2x^2}{21y^3} \]
11. \[ \frac{x^2 + x - 6}{2x^2} \cdot \frac{2x + 8}{x^2 + 7x + 12} \]
12. \[ \frac{144}{4xy} \div \frac{54y^3}{3x^2y} \]
13. \[ \frac{16xy}{3x^2y^3} \div \frac{8y^2}{9x^5y} \]
14. \[ \frac{5x^2 + 10x}{x^2 - x - 6} \div \frac{15x^3 + 45x^2}{x^2 - 9} \]

15. **SKYDIVING** Look back at Example 8 on page 557. Some skydivers wear “wings” to increase their surface area. Suppose a skydiver who is 65 inches tall is wearing wings that add 18\(x^2\) of surface area and an insignificant amount of volume. Calculate the skydiver’s volume to surface area ratio with and without the wings.

**PRACTICE AND APPLICATIONS**

**SIMPLIFYING** If possible, simplify the rational expression.

16. \[ \frac{3x^3}{12x^2 + 9x} \]
17. \[ \frac{x^2 - x - 6}{x^2 + 8x + 16} \]
18. \[ \frac{x^2 - 3x + 2}{x^2 + 5x - 6} \]
19. \[ \frac{x^2 + 2x - 4}{x^2 + x - 6} \]
20. \[ \frac{x^2 - 2x - 3}{x^2 - 7x + 12} \]
21. \[ \frac{3x^2 - 3x - 6}{x^2 - 4} \]
22. \[ \frac{x - 2}{x^3 - 8} \]
23. \[ \frac{x^3 - 27}{x^3 + 3x^2 + 9x} \]
24. \[ \frac{x^2 + 6x + 9}{x^2 - 9} \]
25. \[ \frac{15x^2 - 8x - 18}{-20x^2 + 14x + 2} \]
26. \[ \frac{x^3 - 2x^2 + x - 2}{3x^2 - 3x - 8} \]
27. \[ \frac{x^3 + 3x^2 - 2x - 6}{x^3 + 27} \]

**MULTIPLYING** Multiply the rational expressions. Simplify the result.

28. \[ \frac{4xy^3}{x^2y} \cdot \frac{y}{8x} \]
29. \[ \frac{80x^4}{y^3} \cdot \frac{xy}{5x^2} \]
30. \[ \frac{2x^2 - 10}{x + 1} \cdot \frac{x + 2}{3x^2 - 15} \]
31. \[ \frac{x - 3}{2x - 8} \cdot \frac{6x^2 - 96}{x^2 - 9} \]
32. \[ \frac{x^2 - x - 6}{4x^2} \cdot \frac{x + 1}{x^2 + 5x + 6} \]
33. \[ \frac{2x^2 - 2}{x^2 - 6x - 7} \cdot (x^2 - 10x + 21) \]
34. \[ \frac{x^3 + 5x^2 - x - 5}{x^2 - 25} \cdot (x + 1) \]
35. \[ \frac{x - 3}{-x^3 + 3x^2} \cdot (x^2 + 2x + 1) \]
DIVIDING  Divide the rational expressions. Simplify the result.

36. \( \frac{323y}{y^9} + \frac{8x^4}{y^b} \)
37. \( \frac{2xyz}{x^2} + \frac{6y^3}{3xz} \)
38. \( \frac{3x^2 + x - 2}{x^2 + 3x + 2} + \frac{2x}{x + 2} \)
39. \( \frac{x^2 - 14x + 48}{x^2 - 6x} \)
40. \( \frac{2x^2 - 12x}{x^2 - 7x + 6} + \frac{2x}{3x - 3} \)
41. \( \frac{x^2 + 8x + 16}{x^2 + 2x} + \frac{x^2 + 6x + 8}{x^2 - 4} \)
42. \( \frac{x^2 + 6x - 7}{3x^2} + \frac{x + 7}{6x} \)
43. \( (x^2 + 6x - 27) ÷ \frac{3x^2 + 27x}{x + 5} \)

COMBINED OPERATIONS  Perform the indicated operations. Simplify the result.

44. \( (x - 5) ÷ \frac{x^2 - 11x + 30}{x^2 + 7x + 12} \cdot (x - 6) \)
45. \( \frac{x^2 - x - 12}{8x^2} ÷ \frac{x^3 + 3x^2}{8x^3 - 2x^2} ÷ \frac{4x - 1}{x + 2} \)
46. \( \frac{x^2 + 11x}{x - 2} ÷ \frac{(3x^2 + 6x)}{x + 11} \)
47. \( \frac{2x^2 + x - 15}{2x^2 - 11x - 21} ÷ \frac{(6x + 9)}{2x - 5} ÷ \frac{3x - 21}{3x} \)
48. \( (x^3 + 8) ÷ \frac{x^2 - 2x + 4}{x^2 - 4} ÷ \frac{x^2 - 4}{x - 6} \)
49. \( \frac{x^2 + 12x + 20}{4x^2 - 9} ÷ \frac{6x^3 - 9x^2}{x^3 + 10x^2} ÷ (2x + 3) \)

HEAT GENERATION  In Exercises 50 and 51, use the following information.
Almost all of the energy generated by a long-distance runner is released in the form of heat. The rate of heat generation \( h_g \) and the rate of heat released \( h_r \) for a runner of height \( H \) can be modeled by
\[ h_g = k_1H^3v^2 \]
\[ h_r = k_2H^2 \]
where \( k_1 \) and \( k_2 \) are constants and \( V \) is the runner’s speed.

50. Write the ratio of heat generated to heat released.
51. When the ratio of heat generated to heat released equals 1, how is height related to velocity? Does this mean that a taller or a shorter runner has an advantage?

FARMLAND  In Exercises 52 and 53, use the following information.
From 1987 to 1996, the total acres of farmland \( L \) (in millions) and the total number of farms \( F \) (in hundreds of thousands) in the United States can be modeled by
\[ L = \frac{43.3t + 999}{0.0482t + 1} \text{ and } F = \frac{0.101t^2 + 2.20}{0.0500t^2 + 1} \]
where \( t \) represents the number of years since 1987.  
Source: U.S. Bureau of the Census

52. Write a model for the average number of acres \( A \) per farm as a function of the year.
53. What was the average number of acres per farm in 1993?

WEIGHT IN GOLD  In Exercises 54 and 55, use the following information.
From 1990 to 1996, the price \( P \) of gold (in dollars per ounce) and the weight \( W \) of gold mined (in millions of ounces) in the United States can be modeled by
\[ P = \frac{53.4t^2 - 243t + 385}{0.00146t^3 + 0.122t^2 - 0.586t + 1} \]
\[ W = -0.0112t^3 + 0.193t^2 - 1.17t^3 + 2.82t^2 - 1.76t + 10.4 \]
where \( t \) represents the number of years since 1990.  
Source: U.S. Bureau of the Census

54. Write a model for the total value \( V \) of gold mined as a function of the year.
55. What was the total value of gold mined in the United States in 1994?
56. **GEOMETRY CONNECTION** Use the diagram at the right. Find the ratio of the volume of the rectangular prism to the volume of the inscribed cylinder. Write your answer in simplified form.

57. **MULTI-STEP PROBLEM** The surface area $S$ and the volume $V$ of a tin can are given by $S = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$ where $r$ is the radius of the can and $h$ is the height of the can. One measure of the efficiency of a tin can is the ratio of its surface area to its volume.

   a. Find a general formula (in simplified form) for the ratio $\frac{S}{V}$.

   b. Find the efficiency of a can when $h = 2r$.

   c. Calculate the efficiency of each can.
      - A soup can with $r = 2\frac{5}{8}$ inches and $h = 3\frac{7}{8}$ inches.
      - A 2 pound coffee can with $r = 5\frac{1}{8}$ inches and $h = 6\frac{1}{2}$ inches.
      - A 3 pound coffee can with $r = 6\frac{3}{16}$ inches and $h = 7$ inches.

   d. **Writing** Rank the three cans in part (c) by efficiency (most efficient to least efficient). Explain your rankings.

58. Find two rational functions $f(x)$ and $g(x)$ such that $f(x) \cdot g(x) = x^2$ and
   
   $$f(x) = \frac{(x - 1)^2}{(x + 2)^2}.$$ 

59. Find two rational functions $f(x)$ and $g(x)$ such that $f(x) \cdot g(x) = x - 1$
   and $f(x) = \frac{(x + 1)^2(x - 1)}{x^4}$.

### Mixed Review

**GCFs and LCMs** Find the greatest common factor and least common multiple of each pair of numbers. (Skills Review, p. 908)

- 60. 96, 160
- 61. 120, 165
- 62. 48, 108
- 63. 72, 84
- 64. 238, 51
- 65. 480, 600

**Multiplying Polynomials** Find the product. (Review 6.3 for 9.5)

- 66. $x(x^2 + 7x - 1)$
- 67. $(x + 7)(x - 1)$
- 68. $(x + 10)(x - 3)$
- 69. $(x + 3)(x^2 + 3x + 2)$
- 70. $(2x - 2)(x^3 - 4x^2)$
- 71. $x(x^2 - 4)(5 - 6x^3)$

**BICYCLE DEPRECIATION** In Exercises 72 and 73, use the following information.

You bought a new mountain bike for $800. The value of the bike decreases by about 14% each year. (Review 8.2)

72. Write an exponential decay model for the value of the bike. Use the model to estimate the value after 4 years.

73. Graph the model. Use the graph to estimate when the bike will be worth $300.
Operations with Rational Expressions

You have learned how to simplify rational expressions and how to multiply and divide rational expressions. You can use a graphing calculator to verify the results of these operations numerically and graphically.

**EXAMPLE**

Simplify \( \frac{x^2 + 3x - 10}{x^2 - 5x + 6} \). Use a graphing calculator to verify the results numerically and graphically.

**SOLUTION**

You can simplify the rational expression as follows.

\[
\frac{x^2 + 3x - 10}{x^2 - 5x + 6} = \frac{(x-2)(x+5)}{(x-2)(x-3)} = \frac{x+5}{x-3}
\]

**EXERCISES**

**Simplify the expression. Use a graphing calculator to verify the result numerically and graphically.**

1. \( \frac{y^2 - 3x}{x^2 + x - 12} \)
2. \( \frac{2x^2 - 10x}{x^2 - 4x - 5} \)
3. \( \frac{x^2 + x - 6}{x^2 + 4x + 3} \)

**Perform the indicated operation and simplify. Use a graphing calculator to verify the result numerically and graphically.**

4. \( \frac{x - \frac{1}{2}}{2x^2} \cdot x + 2 \div x - 1 \)
5. \( \frac{2x^2 - 10x}{3x + 3} \div \frac{x - 5}{x + 1} \)
6. \( \frac{x^2 - x - 12}{x^2 + 6x + 6} \cdot \frac{x^2 + 3x + 2}{x^2 + 5x + 6} \)
Addition, Subtraction, and Complex Fractions

GOAL 1 Working with Rational Expressions

As with numerical fractions, the procedure used to add (or subtract) two rational expressions depends upon whether the expressions have like or unlike denominators.

To add (or subtract) two rational expressions with like denominators, simply add (or subtract) their numerators and place the result over the common denominator.

Adding and Subtracting with Like Denominators

Perform the indicated operation.

a. \( \frac{4}{3x} + \frac{5}{3x} \)

b. \( \frac{2x}{x + 3} - \frac{4}{x + 3} \)

Solution

a. \( \frac{4}{3x} + \frac{5}{3x} = \frac{4 + 5}{3x} = \frac{9}{3x} = \frac{3}{x} \) Add numerators and simplify expression.

b. \( \frac{2x}{x + 3} - \frac{4}{x + 3} = \frac{2x - 4}{x + 3} \) Subtract numerators.

To add (or subtract) rational expressions with unlike denominators, first find the least common denominator (LCD) of the rational expressions. Then, rewrite each expression as an equivalent rational expression using the LCD and proceed as with rational expressions with like denominators.

EXAMPLE 2 Adding with Unlike Denominators

Add: \( \frac{5}{6x^2} + \frac{x}{4x^2 - 12x} \)

Solution

First find the least common denominator of \( \frac{5}{6x^2} \) and \( \frac{x}{4x^2 - 12x} \).

It helps to factor each denominator: \( 6x^2 = 6 \cdot x \cdot x \) and \( 4x^2 - 12x = 4 \cdot x \cdot (x - 3) \).

The LCD is \( 12x^2(x - 3) \). Use this to rewrite each expression.

\[
\frac{5}{6x^2} + \frac{x}{4x^2 - 12x} = \frac{5}{6x^2} + \frac{x}{4x(x - 3)} = \frac{5[2(x - 3)]}{6x^2[2(x - 3)]} + \frac{x(3x)}{4x(x - 3)(3x)}
\]

\[= \frac{10x - 30}{12x^2(x - 3)} + \frac{3x^2}{12x^2(x - 3)} \]

\[= \frac{3x^2 + 10x - 30}{12x^2(x - 3)} \]

Why you should learn it

To solve real-life problems, such as modeling the total number of male college graduates in Ex. 47.
EXAMPLE 3  Subtracting With Unlike Denominators

Subtract: \( \frac{x + 1}{x^2 + 4x + 4} - \frac{2}{x^2 - 4} \)

**Solution**

\[
\frac{x + 1}{x^2 + 4x + 4} - \frac{2}{x^2 - 4} = \frac{x + 1}{(x + 2)^2} - \frac{2}{(x - 2)(x + 2)}
\]

\[
= \frac{(x + 1)(x - 2)}{(x + 2)^2(x - 2)} - \frac{2(x + 2)}{(x - 2)(x + 2)(x + 2)}
\]

\[
= \frac{x^2 - x - 2 - (2x + 4)}{(x + 2)^2(x - 2)}
\]

\[
= \frac{x^2 - 3x - 6}{(x + 2)^2(x - 2)}
\]

EXAMPLE 4  Adding Rational Models

The distribution of heights for American men and women aged 20–29 can be modeled by

\[
y_1 = \frac{0.143}{1 + 0.008(x - 70)^4} \quad \text{American men's heights}
\]

\[
y_2 = \frac{0.143}{1 + 0.008(x - 64)^4} \quad \text{American women's heights}
\]

where \( x \) is the height (in inches) and \( y \) is the percent (in decimal form) of adults aged 20–29 whose height is \( x \pm 0.5 \) inches. **Source: Statistical Abstract of the United States**

a. Graph each model. What is the most common height for men aged 20–29? What is the most common height for women aged 20–29?

b. Write a model that shows the distribution of the heights of all adults aged 20–29. Graph the model and find the most common height.

**Solution**

a. From the graphing calculator screen shown at the top right, you can see that the most common height for men is 70 inches (14.3%). The second most common heights are 69 inches and 71 inches (14.2% each). For women, the curve has the same shape, but is shifted to the left so that the most common height is 64 inches. The second most common heights are 63 inches and 65 inches.

b. To find a model for the distribution of all adults aged 20–29, add the two models and divide by 2.

\[
y = \frac{1}{2} \left[ \frac{0.143}{1 + 0.008(x - 70)^4} + \frac{0.143}{1 + 0.008(x - 64)^4} \right]
\]

From the graph shown at the bottom right, you can see that the most common height is 67 inches.
**GOAL 2** SIMPLIFYING COMPLEX FRACTIONS

A complex fraction is a fraction that contains a fraction in its numerator or denominator. To simplify a complex fraction, write its numerator and its denominator as single fractions. Then divide by multiplying by the reciprocal of the denominator.

**EXAMPLE 5** Simplifying a Complex Fraction

Simplify: \( \frac{\frac{2}{x+2} + \frac{2}{x}}{x+2} \)

**Solution**

\[
\frac{2}{x+2} + \frac{2}{x} = \frac{2x + 2}{x(x+2)}
\]

Add fractions in denominator.

\[
= \frac{2}{x+2} \cdot \frac{x(x+2)}{3x+4}
\]

Multiply by reciprocal.

\[
= \frac{2x(x+2)}{(x+2)(3x+4)}
\]

Divide out common factor.

\[
= \frac{2x}{3x + 4}
\]

Write in simplified form.

Another way to simplify a complex fraction is to multiply the numerator and denominator by the least common denominator of every fraction in the numerator and denominator.

**EXAMPLE 6** Simplifying a Complex Fraction

PHOTOGRAPHY The focal length \( f \) of a thin camera lens is given by

\[ f = \frac{1}{\frac{1}{p} + \frac{1}{q}} \]

where \( p \) is the distance between an object being photographed and the lens and \( q \) is the distance between the lens and the film. Simplify the complex fraction.

**Solution**

\[
\frac{1}{\frac{1}{p} + \frac{1}{q}} = \frac{pq}{\frac{1}{p} + \frac{1}{q}} \quad \text{Write equation.}
\]

\[
= \frac{pq}{\frac{1}{p} + \frac{1}{q}} \quad \text{Multiply numerator and denominator by } pq.
\]

\[
= \frac{pq}{q + p} \quad \text{Simplify.}
\]
GUIDED PRACTICE

Vocabulary Check ✓
1. Give two examples of a complex fraction.

Concept Check ✓
2. How is adding (or subtracting) rational expressions similar to adding (or subtracting) numerical fractions?
3. Describe two ways to simplify a complex fraction.
4. Why isn’t \((x + 1)^3\) the LCD of \(\frac{1}{x + 1}\) and \(\frac{1}{(x + 1)^2}\)? What is the LCD?

Skill Check ✓
Perform the indicated operation and simplify.
5. \(\frac{2x}{x + 5} + \frac{7}{x + 5}\)
6. \(\frac{7}{5x} + \frac{8}{3x}\)
7. \(\frac{x}{x - 4} - \frac{6}{x + 3}\)

Simplify the complex fraction.
8. \(\frac{x + 4}{8 + \frac{1}{x}}\)
9. \(\frac{x + 2}{5} - \frac{5}{8 + \frac{4}{x}}\)
10. \(\frac{15}{\frac{2x + 2}{6} - \frac{1}{2}}\)

11. FINANCE For a loan paid back over \(t\) years, the monthly payment is given by \(M = \frac{Pi}{1 - \left(\frac{1}{1 + i}\right)^{12t}}\) where \(P\) is the principal and \(i\) is the annual interest rate. Show that this formula is equivalent to \(M = \frac{P(1 + i)^{12t}}{(1 + i)^{12t} - 1}\).

PRACTICE AND APPLICATIONS

STUDENT HELP
Extra Practice to help you master skills is on p. 953.

OPERATIONS WITH LIKE DENOMINATORS Perform the indicated operation and simplify.
12. \(\frac{7}{6x} + \frac{11}{6x}\)
13. \(\frac{23}{10x^2} - \frac{x}{10x^2}\)
14. \(\frac{4x}{x + 1} - \frac{3}{x + 1}\)
15. \(\frac{5x^2}{x + 8} + \frac{5x}{x + 8}\)
16. \(\frac{6x^2}{x - 2} - \frac{12x}{x - 2}\)
17. \(\frac{x}{x^2 - 5x} - \frac{5}{x^2 - 5x}\)

FINDING LCDS Find the least common denominator.
18. \(\frac{14}{4(x + 1)} \cdot \frac{7}{4}\)
19. \(\frac{4}{21x^2} \cdot \frac{x}{3x^2 - 15x}\)
20. \(\frac{5x + 2}{4x^2 - 1} \cdot \frac{3}{x^2} \cdot \frac{9x}{2x + 1}\)
21. \(\frac{1}{x(x - 6)} \cdot \frac{12}{x^2 - 3x - 18}\)
22. \(\frac{3x + 1}{x(x - 7)} \cdot \frac{3}{x^2 - 6x - 7}\)
23. \(\frac{1}{x^2 - 3x - 28} \cdot \frac{x}{x^2 + 6x + 8}\)

LOGICAL REASONING Tell whether the statement is always true, sometimes true, or never true. Explain your reasoning.
24. The LCD of two rational expressions is the product of the denominators.
25. The LCD of two rational expressions will have a degree greater than or equal to that of the denominator with the higher degree.
Operations with Unlike Denominators Perform the indicated operation(s) and simplify.

26. \( \frac{6}{4x^2} + \frac{2}{5x} \)
27. \( -\frac{4}{7x} - \frac{5}{3x} \)
28. \( \frac{7}{6(x - 2)} - \frac{x + 3}{6x} \)
29. \( \frac{6x + 1}{x^2 - 9} + \frac{4}{x - 3} \)
30. \( \frac{10}{x^2 - 5x - 14} + \frac{2}{x - 7} \)
31. \( \frac{5x - 1}{x^2 + 2x - 8} - \frac{6}{x + 4} \)
32. \( \frac{4x^2}{3x + 5} - \frac{10}{x + 8} \)
33. \( \frac{2 - 5x}{x - 10} + \frac{1}{3x + 2} \)
34. \( \frac{x^2 + x - 3}{x^2 - 12x + 32} + \frac{3x}{x - 8} \)
35. \( \frac{2x + 1}{x^2 + 8x + 16} - \frac{3}{x^2 - 16} \)
36. \( \frac{4x^2}{x + 1} + \frac{5}{2x - 3} - \frac{4}{x} \)

Simplifying Complex Fractions Simplify the complex fraction.

38. \( \frac{x^2 - 5}{6 + \frac{3}{x}} \)
39. \( \frac{\frac{20}{x + 1}}{\frac{1 - \frac{7}{x + 1}}{4}} \)
40. \( \frac{\frac{1}{2x^2 - 2}}{\frac{x}{x + 1} + \frac{2}{x^2 - 2x - 3}} \)
41. \( \frac{\frac{1}{x}}{\frac{x - 1}{x^{-1} + 1}} \)
42. \( \frac{\frac{1}{x^4}}{\frac{x^2 - \frac{2}{x^3 + x^2}}{x + 1}} \)
43. \( \frac{\frac{1}{4x + 3}}{\frac{\frac{5}{x}}{3(4x + 3)}} \)
44. \( \frac{\frac{4}{x^2 - 9} + \frac{2}{x - 3}}{1 + \frac{1}{x + 3} + \frac{1}{x - 3}} \)
45. \( \frac{\frac{1}{x^3 + 64}}{\frac{\frac{5}{x^2 - 16} - \frac{2}{3x^2 + 12x}}{x}} \)
46. \( \frac{\frac{3}{2x^2 + 6x + 18}}{\frac{\frac{x}{x^3 - 27}}{5x}} \)

College Graduates From the 1984–85 school year through the 1993–94 school year, the number of female college graduates \( F \) and the total number of college graduates \( G \) in the United States can be modeled by

\[
F = \frac{-19,600t + 493,000}{-0.0580t + 1} \quad \text{and} \quad G = \frac{7560t^2 + 978,000}{0.00418t^2 + 1}
\]

where \( t \) is the number of school years since the 1984–85 school year. Write a model for the number of male college graduates. Source: U.S. Department of Education

Drug Absorption In Exercises 48–51, use the following information.

The amount \( A \) (in milligrams) of an oral drug, such as aspirin, in a person’s bloodstream can be modeled by

\[
A = \frac{391t^2 + 0.112}{0.218t^4 + 0.991t^2 + 1}
\]

where \( t \) is the time (in hours) after one dose is taken. Source: Drug Disposition in Humans

48. Graph the equation using a graphing calculator.
49. A second dose of the drug is taken 1 hour after the first dose. Write an equation to model the amount of the second dose in the bloodstream.
50. Write and graph a model for the total amount of the drug in the bloodstream after the second dose is taken.
51. About how long after the second dose has been taken is the greatest amount of the drug in the bloodstream?
**ELECTRONICS** In Exercises 52 and 53, use the following information.
If three resistors in a parallel circuit have resistances \( R_1, R_2, \) and \( R_3 \) (all in ohms), then the total resistance \( R_t \) (in ohms) is given by this formula:

\[
R_t = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}
\]

52. Simplify the complex fraction.
53. You have three resistors in a parallel circuit with resistances 6 ohms, 12 ohms, and 24 ohms. What is the total resistance of the circuit?

54. **MULTI-STEP PROBLEM** From 1988 through 1997, the total dollar value \( V \) (in millions of dollars) of the United States sound-recording industry can be modeled by

\[
V = \frac{5783 + 1134t}{1 + 0.025t}
\]

where \( t \) represents the number of years since 1988.

Source: Recording Industry Association of America

a. Calculate the percent change in dollar value from 1988 to 1989.
b. Develop a general formula for the percent change in dollar value from year \( t \) to year \( t + 1 \).
c. Enter the formula into a graphing calculator or spreadsheet. Observe the changes from year to year for 1988 through 1997. Describe what you observe from the data.

**CRITICAL THINKING** In Exercises 55 and 56, use the following expressions.

\[
2 + \frac{1}{1 + \frac{1}{2}}, \quad 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{3}}}, \quad 2 + \frac{1}{1 + \frac{1}{2 + \frac{2}{3 + \frac{3}{4}}}}
\]

55. The expressions form a pattern. Continue the pattern two more times. Then simplify all five expressions.
56. The expressions are getting closer and closer to some value. What is it?

**MIXED REVIEW**

**SOLVING LINEAR EQUATIONS** Solve the equation. (Review 1.3 for 9.6)

57. \( \frac{1}{2}x - 7 = 5 \)  
58. \( 6 - \frac{1}{10}x = -1 \)  
59. \( \frac{3}{4}x + \frac{1}{2} = x - \frac{5}{6} \)

60. \( \frac{3}{8}x + 4 = -8 \)  
61. \( -\frac{1}{12}x - 3 = \frac{5}{2} \)  
62. \( 2 = -\frac{4}{3}x + 10 \)

63. \( -5x - \frac{3}{4}x = \frac{51}{2} \)  
64. \( 2x + \frac{7}{8}x = -23 \)  
65. \( x = 12 + \frac{5}{6}x \)

**SOLVING QUADRATIC EQUATIONS** Solve the equation. (Review 5.2, 5.3 for 9.6)

66. \( x^2 - 5x - 24 = 0 \)  
67. \( 5x^2 - 8 = 4(x^2 + 3) \)  
68. \( 6x^2 + 13x - 5 = 0 \)

69. \( 3(x - 5)^2 = 27 \)  
70. \( 2(x + 7)^2 - 1 = 49 \)  
71. \( 2x(x + 6) = 7 - x \)

9.5 Addition, Subtraction, and Complex Fractions
Solving Rational Equations

**GOAL 1** SOLVING A RATIONAL EQUATION

To solve a rational equation, multiply each term on both sides of the equation by the LCD of the terms. Simplify and solve the resulting polynomial equation.

**EXAMPLE 1** An Equation with One Solution

Solve: \( \frac{4}{x} + \frac{5}{2} = -\frac{11}{x} \)

**Solution**

The least common denominator is \( 2x \).

\[
\frac{4}{x} + \frac{5}{2} = -\frac{11}{x} \quad \text{Write original equation.}
\]

\[
2x \left( \frac{4}{x} + \frac{5}{2} \right) = 2x \left( -\frac{11}{x} \right) \quad \text{Multiply each side by } 2x.
\]

\[
8 + 5x = -22 \quad \text{Simplify.}
\]

\[
5x = -30 \quad \text{Subtract 8 from each side.}
\]

\[
x = -6 \quad \text{Divide each side by 5.}
\]

The solution is \(-6\). Check this in the original equation.

**EXAMPLE 2** An Equation with an Extraneous Solution

Solve: \( \frac{-5x}{x-2} = 7 + \frac{10}{x-2} \)

**Solution**

The least common denominator is \( x - 2 \).

\[
\frac{-5x}{x-2} = 7 + \frac{10}{x-2} \quad \text{Write original equation.}
\]

\[
(x - 2) \cdot \frac{-5x}{x-2} = (x - 2) \cdot 7 + (x - 2) \cdot \frac{10}{x-2} \quad \text{Multiply each side by } x - 2.
\]

\[
5x = 7(x - 2) + 10
\]

\[
5x = 7x - 4
\]

\[
x = 2
\]

The solution appears to be \(2\). After checking it in the original equation, however, you can conclude that \(2\) is an extraneous solution because it leads to division by zero. So, the original equation has no solution.
**EXAMPLE 3**  An Equation with Two Solutions

Solve: \( \frac{4x + 1}{x + 1} = \frac{12}{x^2 - 1} + 3 \)

**SOLUTION**

Write each denominator in factored form. The LCD is \((x + 1)(x - 1)\).

\[
\begin{align*}
\frac{4x + 1}{x + 1} &= \frac{12}{(x + 1)(x - 1)} + 3 \\
(x + 1)(x - 1) \cdot \frac{4x + 1}{x + 1} &= (x + 1)(x - 1) \cdot \frac{12}{(x + 1)(x - 1)} + (x + 1)(x - 1) \cdot 3 \\
(x - 1)(4x + 1) &= 12 + 3(x + 1)(x - 1) \\
4x^2 - 3x - 1 &= 12 + 3x^2 - 3 \\
x^2 - 3x - 10 &= 0 \\
(x + 2)(x - 5) &= 0 \\
x + 2 &= 0 \quad \text{or} \quad x - 5 &= 0 \\
x &= -2 \quad \text{or} \quad x = 5
\end{align*}
\]

The solutions are \(-2\) and 5. Check these in the original equation.

You can use **cross multiplying** to solve a simple rational equation for which each side of the equation is a single rational expression.

**EXAMPLE 4**  Solving an Equation by Cross Multiplying

Solve: \( \frac{-2}{x^2 - x} = \frac{1}{x - 1} \)

**SOLUTION**

\[
\begin{align*}
\frac{-2}{x^2 - x} &= \frac{1}{x - 1} \\
2(x - 1) &= 1(x^2 - x) \\
2x - 2 &= x^2 - x \\
0 &= x^2 - 3x + 2 \\
0 &= (x - 2)(x - 1) \\
x &= 2 \text{ or } x = 1
\end{align*}
\]

The solutions appear to be 2 and 1. After checking them in the original equation, however, you see that the only solution is 2. The apparent solution \(x = 1\) is extraneous. A graphic check shows that the graphs of the left and right sides of the equation, \( y = \frac{-2}{x^2 - x} \) and \( y = \frac{1}{x - 1} \), intersect only at \( x = 2 \). At \( x = 1 \), the graphs have a common vertical asymptote.
**EXAMPLE 5**  **Writing and Using a Rational Model**

**CHEMISTRY** You have 0.2 liter of an acid solution whose acid concentration is 16 moles per liter. You want to dilute the solution with water so that its acid concentration is only 12 moles per liter. How much water should you add to the solution?

**SOLUTION**

**VERBAL MODEL**

- Concentration of new solution = \(12\) (moles per liter)
- Moles of acid in original solution = \(16(0.2)\) (moles)
- Volume of original solution = \(0.2\) (liters)
- Volume of water added = \(x\) (liters)

\[
12 = \frac{16(0.2)}{0.2 + x}
\]

Write equation.

\[
12(0.2 + x) = 16(0.2)
\]

Multiply each side by \(0.2 + x\).

\[
2.4 + 12x = 3.2
\]

Simplify.

\[
12x = 0.8
\]

Subtract 2.4 from each side.

\[
x = 0.067
\]

Divide each side by 12.

You should add about 0.067 liter, or 67 milliliters, of water.

**EXAMPLE 6**  **Using a Rational Model**

**RODEOS** From 1980 through 1997, the total prize money \(P\) (in millions of dollars) at Professional Rodeo Cowboys Association events can be modeled by

\[
P = \frac{380t + 5}{-t^2 + 31t + 1}
\]

where \(t\) represents the number of years since 1980. During which year was the total prize money about $20 million?  

**SOLUTION**

Use a graphing calculator to graph the equation

\[y = \frac{380x + 5}{-x^2 + 31x + 1}\]

Then graph the line \(y = 20\). Use the *Intersect* feature to find the value of \(x\) that gives a \(y\)-value of 20. As shown at the right, this value is \(x = 12\). So, the total prize money was about $20 million 12 years after 1980, in 1992.
**GUIDED PRACTICE**

**Vocabulary Check ✓**
1. Give an example of a rational equation that can be solved using cross multiplication.

**Concept Check ✓**
2. A student solved the equation \( \frac{2}{x - 3} = \frac{x}{x - 3} \) and got the solutions 2 and 3. Which, if either, of these is extraneous? Explain how you know.

3. Describe two methods that can be used to solve a rational equation. Which method can always be used? Why?

4. Solve the equation \( \frac{1}{x} = \frac{2}{x^2} \). Check the apparent solutions graphically. Explain how a graph can help you identify actual and extraneous solutions.

**Skill Check ✓**

Solve the equation using any method. Check each solution.

5. \( \frac{7}{x} + \frac{3}{4} = \frac{5}{x} \)
6. \( \frac{x - 2}{6} = \frac{x - 2}{x - 1} \)
7. \( 3x + \frac{x}{3} = 5 \)

8. \( \frac{x}{x - 3} = 2 - \frac{2}{x - 3} \)
9. \( \frac{5}{x - 3} = \frac{2x}{x^2 - 9} \)
10. \( \frac{5x}{x - 1} + 5 = \frac{15}{x - 1} \)

11. \( \frac{2x}{x + 3} = \frac{3x}{x - 3} \)
12. \( \frac{2x}{x - 4} = \frac{8}{x - 4} + 3 \)
13. \( \frac{2x}{2x + 4} = \frac{3x}{x + 2} \)

14. **Basketball Statistics** So far in the basketball season you have made 12 free throws out of the 20 free throws you have attempted, for a free-throw shooting percentage of 60%. How many consecutive free-throw shots would you have to make to raise your free-throw shooting percentage to 80%?

**PRACTICE AND APPLICATIONS**

**Checking Solutions** Determine whether the given x-value is a solution of the equation.

15. \( \frac{2x - 3}{x + 3} = \frac{3x}{x + 4}; x = -1 \)
16. \( \frac{x}{2x + 1} = \frac{5}{4 - x}; x = -1 \)
17. \( \frac{4x - 3}{x - 4} + 1 = \frac{x}{x - 3}; x = 2 \)
18. \( \frac{3x}{x - 6} = 5 + \frac{18}{x - 6}; x = 6 \)
19. \( \frac{x}{x - 3} = \frac{6}{x - 3}; x = 6 \)
20. \( \frac{2}{x(x + 2)} + \frac{3}{x} = \frac{4}{x - 2}; x = 2 \)

**Least Common Denominator** Solve the equation by using the LCD. Check each solution.

21. \( \frac{3}{2} + \frac{1}{x} = 2 \)
22. \( \frac{3}{x} + x = 4 \)
23. \( \frac{3}{2x} - \frac{9}{2} = 6x \)

24. \( \frac{8}{x + 2} + \frac{8}{2} = 5 \)
25. \( \frac{3x}{x + 1} + \frac{6}{2x} = \frac{7}{x} \)
26. \( \frac{2}{3x} + \frac{2}{3} = \frac{8}{x + 6} \)

27. \( \frac{6x}{x + 4} + 4 = \frac{2x + 2}{x - 1} \)
28. \( \frac{x - 3}{x - 4} + 4 = \frac{3x}{x} \)
29. \( \frac{7x + 1}{2x + 5} + 1 = \frac{10x - 3}{3x} \)

30. \( \frac{10}{x^2 - 2x} + \frac{4}{x} = \frac{5}{x - 2} \)
31. \( \frac{4(x - 1)}{x - 1} = \frac{2x - 2}{x + 1} \)
32. \( \frac{2(x + 7)}{x + 4} - 2 = \frac{2x + 20}{2x + 8} \)
**CROSS MULTIPLYING**  Solve the equation by cross multiplying. Check each solution.

33. \( \frac{3}{4x} = \frac{5}{x + 2} \) 
34. \( -\frac{3}{x + 1} = \frac{4}{x - 1} \) 
35. \( \frac{x}{x^2 - 8} = \frac{2}{x} \) 
36. \( \frac{x}{2x + 7} = \frac{x - 5}{x - 1} \) 
37. \( -\frac{2}{x - 1} = \frac{x - 8}{x + 1} \) 
38. \( \frac{2(x - 2)}{x^2 - 10x + 16} = \frac{2}{x + 2} \) 
39. \( \frac{8(x - 1)}{x^2 - 4} = \frac{4}{x - 2} \) 
40. \( \frac{x^2 - 3}{x + 2} = \frac{x - 3}{2} \) 
41. \( -\frac{1}{x - 3} = \frac{x - 4}{x^2 - 27} \)

**CHOOSING A METHOD**  Solve the equation using any method. Check each solution.

42. \( \frac{x - 2}{x + 2} = \frac{3}{x} \) 
43. \( \frac{3}{x + 2} = \frac{6}{x - 1} \) 
44. \( \frac{3x}{x + 1} = \frac{12}{x^2 - 1} + 2 \) 
45. \( \frac{3x + 6}{x^2 - 4} = \frac{x + 1}{x - 2} \) 
46. \( \frac{x - 4}{x} = \frac{6}{x^2 - 3x} \) 
47. \( \frac{2x}{4 - x} = \frac{x^2}{x - 4} \) 
48. \( \frac{2x}{x - 3} = \frac{3x}{x^2 - 9} + 2 \) 
49. \( \frac{x}{2x - 6} = \frac{2}{x - 4} \) 
50. \( \frac{2}{x + 1} + \frac{x - 1}{x} = \frac{2}{x^2 - 1} \)

**LOGICAL REASONING**  In Exercises 51–53, \( a \) is a nonzero real number. Tell whether the algebraic statement is *always true*, *sometimes true*, or *never true*. Explain your reasoning.

51. For the equation \( \frac{1}{x - a} = \frac{x}{x - a} \), \( x = a \) is an extraneous solution.
52. The equation \( \frac{3}{x - a} = \frac{x}{x - a} \) has exactly one solution.
53. The equation \( \frac{1}{x - a} = \frac{2}{x + a} + \frac{2a}{x^2 - a^2} \) has no solution.

54. **Football Statistics**  At the end of the 1998 season, the National Football League’s all-time leading passer during regular season play was Dan Marino with 4763 completed passes out of 7989 attempts. In his debut 1998 season, Peyton Manning made 326 completed passes out of 575 attempts. How many consecutive completed passes would Peyton Manning have to make to equal Dan Marino’s pass completion percentage?

**Data Update** of National Football League data at www.mcdougallittell.com

55. **Phone Cards**  A telephone company offers you an opportunity to sell prepaid, 30 minute long-distance phone cards. You will have to pay the company a one-time setup fee of $200. Each phone card will cost you $5.70. How many cards would you have to sell before your average total cost per card falls to $8?

56. **River Current**  It takes a paddle boat 53 minutes to travel 5 miles up a river and 5 miles back, going at a steady speed of 12 miles per hour (with respect to the water). Find the speed of the current.

57. **Biology > Connection**  The number \( f \) of flies eaten by a praying mantis in 8 hours can be modeled by

\[
 f = \frac{26.6d}{d + 0.0017}
\]

where \( d \) is the density of flies available (in flies per cubic centimeter). Approximate the density of flies (in flies per cubic meter) when a praying mantis eats 15 flies in 8 hours. (Hint: There are 1,000,000 cm\(^3\) in 1 m\(^3\).)  

*Source: Biology by Numbers*
**Fuel Efficiency** In Exercises 58 and 59, use the following information. The cost of fueling your car for one year can be calculated using this equation:

\[
\text{Fuel cost for one year} = \frac{\text{Miles driven} \times \text{Price per gallon of fuel}}{\text{Fuel efficiency rate}}
\]

58. Last year you drove 9000 miles, paid $1.10 per gallon of gasoline, and spent a total of $412.50 on gasoline. What is the fuel efficiency rate of your car?

59. How much would you have saved if your car’s fuel efficiency rate were 25 miles per gallon?

**Quantitative Comparison** In Exercises 60 and 61, choose the statement below that is true about the given quantities.

- A. The quantity in column A is greater.
- B. The quantity in column B is greater.
- C. The two quantities are equal.
- D. The relationship cannot be determined from the given information.

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>60. The solution of (\frac{x^3 + 1}{x} = 2x^2)</td>
<td>The solution of (\frac{-2}{x + 3} = \frac{4}{x - 2})</td>
</tr>
<tr>
<td>61. The solution of (\frac{1}{x} + 3 = \frac{9}{2x})</td>
<td>The solution of (\frac{1}{2} + \frac{3}{x} = \frac{43}{14})</td>
</tr>
</tbody>
</table>

**Challenge**

62. **Science Connection** You have 0.5 liter of an acid solution whose acid concentration is 16 moles per liter. To decrease the acid concentration to 12 moles per liter, you plan to add a certain amount of a second acid solution whose acid concentration is only 10 moles per liter. How many liters of the second acid solution should you add?

**Mixed Review**

**Slopes of Lines** Find the slope of a line parallel to the given line and the slope of a line perpendicular to the given line. (Review 2.4 for 10.1)

63. \(y = x + 3\)  
64. \(y = 3x - 4\)  
65. \(y = -\frac{2}{3}x + 15\)

66. \(y + 3 = 3x + 2\)  
67. \(2y - x = 7\)

**Properties of Square Roots** Simplify the expression. (Review 5.3 for 10.1)

69. \(\sqrt{48}\)  
70. \(\sqrt{18}\)  
71. \(\sqrt{108}\)  
72. \(\sqrt{432}\)

73. \(\sqrt{6} \cdot \sqrt{45}\)  
74. \(\frac{\sqrt{16}}{\sqrt{72}}\)  
75. \(\sqrt{75} \cdot \sqrt{3}\)  
76. \(\frac{\sqrt{8}}{\sqrt{49}}\)

77. **Geology** You can find the pH of a soil by using the formula

\[
pH = -\log [H^+]
\]

where \([H^+]\) is the soil’s hydrogen ion concentration (in moles per liter). Find the pH of a layer of soil that has a hydrogen ion concentration of \(1.6 \times 10^{-7}\) moles per liter. (Review 8.4)
Perform the indicated operation and simplify. (Lessons 9.4 and 9.5)

1. \( \frac{3x^3y}{2xy^2} \cdot \frac{10x^4y^2}{9x} \)

2. \( \frac{x^2 - 3x - 40}{5x} \div (x + 5) \)

3. \( \frac{18x}{x^2 - 5x - 36} + \frac{2x}{x + 4} \)

4. \( \frac{8x^2}{25x^2 - 36} - \frac{1}{10x + 12} \)

Simplify the complex fraction. (Lesson 9.5)

5. \( \frac{8}{x} + 11 \)

6. \( \frac{36 - \frac{1}{x^2}}{\frac{1}{6x^2} - 6} \)

7. \( \frac{2}{x^2 - 1} - \frac{1}{x + 1} \)

8. \( \frac{1}{x - 5} - \frac{x}{x^2 - 25} \)

9. **Average Cost** You bought a potholder weaving frame for $10. A bag of potholder material costs $4 and contains enough material to make a dozen potholders. How many dozens of potholders must you make before your average total cost per dozen falls to $4.50? (Lesson 9.6)

---

**Math & History**

**Deep Water Diving**

**THEN**

In 1530 the invention of the diving bell provided the first effective means of breathing underwater. Like many other diving devices, a diving bell uses air that is compressed by the pressure of the water. Because oxygen under high pressure (at great depths) can have toxic effects on the body, the percent of oxygen in the air must be adjusted. The recommended percent \( p \) of oxygen (by volume) in the air that a diver breathes is

\[
p = \frac{660}{d + 33}
\]

where \( d \) is the depth (in feet) at which the diver is working.

1. Graph the equation.
2. At what depth is the recommended percent of oxygen 5%?
3. What value does the recommended percent of oxygen approach as a diver’s depth increases?

**NOW**

Today diving technology makes it easier for scientists like Dr. Sylvia Earle to study ocean life. Using one-person submarines, Earle has undertaken a five-year study of marine sanctuaries.

**APPLICATION LINK**

www.mcdougallittell.com

Sylvia Earle, marine biologist.
WHAT did you learn?
Write and use variation models.
• inverse variation (9.1)
• joint variation (9.1)

Graph rational functions.
• simple rational functions (9.2)
• general rational functions (9.3)

Perform operations with rational expressions.
• multiply and divide (9.4)
• add and subtract (9.5)

Simplify complex fractions. (9.5)

Solve rational equations. (9.6)

Use rational models to solve real-life problems. (9.1–9.6)

WHY did you learn it?

Find the speed of a whirlpool’s current. (p. 535)
Find the heat loss through a window. (p. 539)

Describe the frequency of an approaching ambulance siren. (p. 545)
Find the energy expenditure of a parakeet. (p. 551)

Compare the velocities of two skydivers. (p. 557)
Write a model for the number of male college graduates in the United States. (p. 566)

Write a simplified model for the focal length of a camera lens. (p. 564)

Find the amount of water to add when diluting an acid solution. (p. 570)

Find the year in which a certain amount of rodeo prize money was earned. (p. 570)

How does Chapter 9 fit into the BIGGER PICTURE of algebra?

In Chapter 9 you studied rational functions. A rational function is the ratio of two polynomial functions, which you studied in Chapter 2 (linear functions), Chapter 5 (quadratic functions), and Chapter 6 (polynomial functions).

A hyperbola is the graph of one important type of rational function. In the next chapter you will learn more about hyperbolas, parabolas, circles, and ellipses, which together are called the conic sections.

STUDY STRATEGY
How did you make and use a dictionary of functions?

Here is an example of one entry in a dictionary of functions, following the Study Strategy on page 532.
Chapter Review

VOCABULARY

- inverse variation, p. 534
- constant of variation, p. 534
- joint variation, p. 536
- rational function, p. 540
- hyperbola, p. 540
- branches of a hyperbola, p. 540
- simplified form of a rational expression, p. 554
- complex fraction, p. 564
- cross multiplying, p. 569

9.1 INVERSE AND JOINT VARIATION

EXAMPLES You can write an inverse or joint variation equation using a general equation for the variation and given values of the variables.

Inverse variation: \( x = 5, y = 4 \)

\[ y = \frac{k}{x} \quad \text{y varies inversely with } x. \]

\[ 4 = \frac{k}{5} \quad \text{Substitute for } x \text{ and } y. \]

\[ 20 = k \quad \text{Solve for } k. \]

The inverse variation equation is \( y = \frac{20}{x} \).

Joint variation: \( x = 3, y = 8, z = 30 \)

\[ z = kxy \quad z \text{ varies jointly with } x \text{ and } y. \]

\[ 30 = k(3)(8) \quad \text{Substitute for } x, y, \text{ and } z. \]

\[ 30 = 24k \quad \text{Multiply.} \]

\[ k = \frac{30}{24} = \frac{5}{4} \quad \text{Solve for } k. \]

The joint variation equation is \( z = \frac{5}{4}xy \).

The variables \( x \) and \( y \) vary inversely. Use the given values to write an equation relating \( x \) and \( y \). Then find \( y \) when \( x = 2 \).

1. \( x = 1, y = 5 \)  
2. \( x = 15, y = \frac{2}{3} \)  
3. \( x = \frac{1}{4}, y = 8 \)  
4. \( x = -2, y = 2 \)

The variable \( z \) varies jointly with \( x \) and \( y \). Use the given values to write an equation relating \( x, y, \) and \( z \). Then find \( z \) when \( x = 5 \) and \( y = -6 \).

5. \( x = 1, y = 12, z = 4 \)  
6. \( x = 6, y = 8, z = -6 \)  
7. \( x = \frac{3}{4}, y = 4, z = 9 \)

9.2 GRAPHING SIMPLE RATIONAL FUNCTIONS

EXAMPLE 1 To graph \( y = \frac{1}{x + 2} + 3 \), note that the asymptotes are \( x = -2 \) and \( y = 3 \). Plot two points to the left of the vertical asymptote, such as \((-3, 2)\) and \((-4, 2.5)\), and two points to the right, such as \((-1, 4)\) and \((0, 3.5)\). Use the asymptotes and plotted points to draw the branches of the hyperbola. The domain is all real numbers except \(-2\), and the range is all real numbers except \(3\).
EXAMPLE 2 To graph \( y = \frac{x + 1}{x - 3} \), note that when the denominator equals zero, \( x = 3 \). So the vertical asymptote is \( x = 3 \). The horizontal asymptote, which occurs at the ratio of the x-coefficients, is \( y = 1 \). Plot some points to the left and right of the vertical asymptote. Use the asymptotes and plotted points to draw the branches of the hyperbola. The domain is all real numbers except 3, and the range is all real numbers except 1.

Graph the function. State the domain and range.

8. \( y = \frac{3}{x - 5} \) 9. \( y = \frac{1}{x + 4} + 2 \) 10. \( y = \frac{-6x}{x + 2} \) 11. \( y = \frac{2x + 5}{x - 1} \)

EXAMPLE To graph \( y = \frac{3x^2}{x + 2} \), note that the numerator has 0 as its only real zero, so the graph has one x-intercept at \( (0, 0) \). The only zero of the denominator is \( -2 \), so the only vertical asymptote is \( x = -2 \). The degree of the numerator is greater than the degree of the denominator, so there is no horizontal asymptote.

Graph the function.

12. \( y = \frac{3x^2 + 1}{x^2 - 1} \) 13. \( y = \frac{x^3}{10} \) 14. \( y = \frac{x}{x^2 - 4} \) 15. \( y = \frac{3x^2 - 4x + 1}{x^2 - 2x - 3} \)

MULTIPLYING AND DIVIDING RATIONAL EXPRESSIONS

EXAMPLE Dividing rational expressions is like dividing numerical fractions.

\[
\frac{x^2 - 9}{5(x + 2)} \div \frac{x - 3}{5(x^2 - 4)} = \frac{x^2 - 9}{5(x + 2)} \cdot \frac{5(x^2 - 4)}{x - 3} \\
= \frac{(x + 3)(x - 3)(5)(x + 2)(x - 2)}{5(x + 2)(x - 3)} \\
= (x + 3)(x - 2)
\]

Perform the indicated operation(s). Simplify the result.

16. \( \frac{x^2 - 3x}{4x^2 - 8x} \cdot (4x^2 - 16) \) 17. \( \frac{5x}{x + 6} \cdot \frac{x^2 - 9}{x} \) 18. \( \frac{x^2 - 2x - 3}{x + 1} \div \frac{x^2 + x - 12}{x^2} - 1 \)
9.5 **ADDITION, SUBTRACTION, AND COMPLEX FRACTIONS**

**EXAMPLES** You can use the LCD to add or subtract rational expressions.

\[ \frac{3}{x-3} - \frac{5}{x+2} = \frac{3(x+2)}{(x-3)(x+2)} - \frac{5(x-3)}{(x-3)(x+2)} \]

\[ = \frac{3(x+2) - 5(x-3)}{(x-3)(x+2)} \]

\[ = \frac{3x+6 - 5x + 15}{(x-3)(x+2)} \]

\[ = -\frac{2x+21}{(x-3)(x+2)} \]

To simplify a complex fraction, divide the numerator by the denominator.

\[ \frac{2 + 4}{2x + 1} = \frac{2 + 4x}{2x + 1} \]

\[ = \frac{2 + 4x}{2x + 1} \cdot \frac{5x^2}{2x + 1} = \frac{2(1 + 2x)(5x^2)}{x(2x + 1)} = 10x \]

**Perform the indicated operation(s) and simplify.**

19. \( \frac{5}{x^2(x-2)} + \frac{x}{x-2} \)

20. \( \frac{x+5}{x-5} - \frac{3}{x+5} \)

21. \( \frac{x-2}{5x(x-1)} + \frac{1}{x-1} - \frac{3x+2}{x^2+4x-5} \)

22. \( \frac{x+3}{1 + \frac{x}{3}} \)

23. \( \frac{\frac{x}{2} - 4}{9 + \frac{2}{x}} \)

24. \( \frac{\frac{1}{x+1} + \frac{1}{x-1}}{\frac{x}{x+1}} \)

25. \( \frac{\frac{4}{5-x}}{\frac{2}{5-x} + \frac{1}{3-x-15}} \)

9.6 **SOLVING RATIONAL EQUATIONS**

**EXAMPLES** You can solve rational equations by multiplying each side of the equation by the LCD of the terms. If each side of the equation is a single rational expression, you can use cross multiplying. Check for extraneous solutions.

\[ \frac{4}{x} + \frac{3}{2x} = 11 \]

\[ (2x)^\frac{4}{x} + (2x)^\frac{3}{2x} = (2x)11 \]

\[ 8 + 3 = 22x \]

\[ x = \frac{1}{2} \]

\[ \frac{2}{3x+6} = \frac{x+2}{x^2-10} \]

\[ 2(x^2 - 10) = (x + 2)(3x + 6) \]

\[ 2x^2 - 20 = 3x^2 + 12x + 12 \]

\[ 0 = x^2 + 12x + 32 \]

\[ 0 = (x + 8)(x + 4) \]

\[ x = -8 \text{ or } x = -4 \]

**Solve the equation using any method. Check each solution.**

26. \( \frac{x}{x-1} = \frac{2x+10}{x+11} \)

27. \( \frac{x+3}{x} - 1 = \frac{1}{x-1} \)

28. \( \frac{2}{x-2} - \frac{2x}{3} = \frac{x-3}{3} \)

29. \( \frac{3x+2}{x+1} = 2 - \frac{2x+3}{x+1} \)

30. \( \frac{2}{x-6} = -\frac{5}{x+1} \)

31. \( 1 + \frac{3}{x-3} = \frac{4}{x^2 - 9} \)
The variables $x$ and $y$ vary inversely. Use the given values to write an equation relating $x$ and $y$. Then find $y$ when $x = 3$.

1. $x = -4, y = 9$
2. $x = \frac{1}{2}, y = 5$
3. $x = 12, y = \frac{2}{3}$
4. $x = 6, y = -1$

The variable $z$ varies jointly with $x$ and $y$. Use the given values to write an equation relating $x$, $y$, and $z$. Then find $z$ when $x = -2$ and $y = 4$.

5. $x = 5, y = 4, z = 2$
6. $x = -3, y = 2, z = 18$
7. $x = \frac{1}{3}, y = \frac{3}{4}, z = \frac{5}{2}$

Graph the function.

8. $y = \frac{-1}{x + 1} - 2$
9. $y = \frac{4}{x - 2}$
10. $y = \frac{x}{2x + 5}$
11. $y = \frac{4x - 3}{x - 4}$
12. $y = \frac{6}{x^2 + 4}$
13. $y = \frac{-3x^2}{2x - 1}$
14. $y = \frac{x^2 - 2}{x^2 - 9}$
15. $y = \frac{x^2 - 2x + 15}{x + 1}$

Perform the indicated operation. Simplify the result.

16. $\frac{x^2 - 4}{x + 3} \cdot \frac{x^2 + 4x + 3}{2x - 4}$
17. $\frac{4x - 8}{x^2 - 3x + 2} \div \frac{3x - 6}{x - 1}$
18. $\frac{x + 4}{x^2 - 25} \cdot (x^2 + 3x - 10)$
19. $\frac{5}{6x} + \frac{7}{18x}$
20. $\frac{x - 1}{x - 2} - \frac{x - 4}{x + 1}$
21. $\frac{3x}{x^2 - 10x + 21} + \frac{5}{x - 3}$

Simplify the complex fraction.

22. $\frac{1 + \frac{3}{x}}{2 - \frac{5}{x^2}}$
23. $\frac{\frac{4 + x}{10}}{\frac{x^2 - 16}{8}}$
24. $\frac{\frac{2}{x - 1} + \frac{5}{x}}{\frac{3}{3}}$
25. $\frac{\frac{36}{1}}{\frac{1}{x} + \frac{7}{2x}}$

Solve the equation using any method. Check each solution.

26. $\frac{9}{x} + \frac{11}{5} = \frac{31}{x}$
27. $\frac{-15}{x} = x + \frac{16}{4}$
28. $\frac{8}{x + 3} = \frac{5}{x - 3}$
29. $\frac{4x}{x + 3} = \frac{37}{x^2 - 9} - 3$

30. **Science Connection** A lever pivots on a support called a *fulcrum*. For a balanced lever, the distance $d$ (in feet) an object is from the fulcrum varies inversely with the object’s weight $w$ (in pounds). An object weighing 140 pounds is placed 6 feet from a fulcrum. How far from the fulcrum must a 112 pound object be placed to balance the lever?

31. **Geometry Connection** A sphere with radius $r$ is inscribed in a cube as shown. Find the ratio of the volume of the cube to the volume of the sphere. Write your answer in simplified form.

32. **Starting a Business** You start a small bee-keeping business, spending $500 for equipment and bees. You figure it will cost $1.25 per pound to collect, clean, bottle, and label the honey. How many pounds of honey must you produce before your average cost per pound is $1.79?
Chapter Standardized Test

1. **Multiple Choice** The variable \( x \) varies inversely with \( y \). When \( x = 6 \), \( y = 6.5 \). Which equation relates \( x \) and \( y \)?
   - A. \( xy = 39 \)
   - B. \( xy = 11.5 \)
   - C. \( xy = \frac{1}{2} \)
   - D. \( y = \frac{1}{2}x \)
   - E. \( y = 39x \)

2. **Multiple Choice** The variable \( z \) varies jointly with \( x \) and \( y \). When \( x = 6 \) and \( y = \frac{1}{3} \), \( z = 30 \). Which equation relates \( x \), \( y \), and \( z \)?
   - A. \( z = 30xy \)
   - B. \( 30 = xyz \)
   - C. \( z = 15xy \)
   - D. \( z = \frac{1}{30}xy \)
   - E. \( z = \frac{1}{15}xy \)

3. **Multiple Choice** Which function is graphed?

   - A. \( y = \frac{10}{x + 5} - 3 \)
   - B. \( y = \frac{10}{x - 5} - 3 \)
   - C. \( y = \frac{10}{x + 5} + 3 \)
   - D. \( y = \frac{10}{x - 5} + 3 \)
   - E. \( y = \frac{10}{x - 5} \)

4. **Multiple Choice** What is the quotient \( \frac{x + 2}{\frac{x^2 - 9x - 22}{x^2 - 121}} \)?
   - A. \( x + 11 \)
   - B. \( \frac{x + 11}{x + 2} \)
   - C. \( \frac{x + 2}{x + 11} \)
   - D. \( \frac{x + 2}{x - 11} \)
   - E. \( x + 2 \)

5. **Multiple Choice** What are all the solutions of the equation \( \frac{-10}{x - 9} = \frac{x}{2} \)?
   - A. \(-4, -5\)
   - B. \(4, -5\)
   - C. \(4\)
   - D. \(5\)
   - E. \(4, 5\)

6. **Multiple Choice** Which function is graphed?
   - A. \( y = \frac{-3x^2}{x^2 - 25} \)
   - B. \( y = \frac{-3x^2}{x^2 - 16} \)
   - C. \( y = \frac{-3x^2}{x^2 - 25} \)
   - D. \( y = \frac{3x^2}{x^2 - 25} \)
   - E. \( y = \frac{-3x^2}{x^2 + 25} \)

7. **Multiple Choice** What is the difference \( \frac{8x - 3}{x^2 + 2x - 35} - \frac{7}{x^2 - 25} \)?
   - A. \( \frac{2(4x^2 + 15x - 17)}{(x^2 + 2x - 35)(x + 5)} \)
   - B. \( \frac{2(4x^2 + 15x + 32)}{(x^2 + 2x - 35)(x + 5)} \)
   - C. \( \frac{2(4x^2 + 15x + 17)}{(x^2 + 2x - 35)(x + 5)} \)
   - D. \( \frac{2(4x^2 + 15x - 32)}{(x^2 + 2x - 35)(x^2 - 25)} \)
   - E. \( \frac{2(4x^2 + 15x - 32)}{(x^2 + 2x - 35)(x^2 - 25)} \)

8. **Multiple Choice** What is the simplified form of the following complex fraction?
   - A. \( \frac{10}{x + 1} \)
   - B. \( \frac{20}{x + 7} \)
   - C. \( \frac{10}{x + 7} \)
   - D. \( \frac{10(x + 7)}{x + 1} \)
   - E. \( 20 \)
9. **Quantitative Comparison** Choose the statement that is true about the given quantities.

- **A** The quantity in column A is greater.
- **B** The quantity in column B is greater.
- **C** The two quantities are equal.
- **D** The relationship cannot be determined from the given information.

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>The solution of ( \frac{x - 4}{x + 1} = \frac{7}{2} )</td>
<td>The solution of ( \frac{5x - 8}{3} = \frac{1}{12x} )</td>
</tr>
</tbody>
</table>

10. **Multi-Step Problem** For parts (a)–(d), graph the function and identify the point at which the horizontal and vertical asymptotes intersect.

a. \( y = \frac{2}{x} \)  
   b. \( y = \frac{2}{x - 1} + 3 \)  
   c. \( y = \frac{2}{x - 1} - 3 \)  
   d. \( y = \frac{2}{x + 1} + 3 \)

e. Use your answers to parts (a)–(d) to predict the point of intersection of the asymptotes of the graph of \( y = \frac{2}{x + 1} - 3 \). Check your prediction by graphing.

f. **Critical Thinking** Generalize your results for any function of the form \( y = \frac{a}{x - h} + k \).

11. **Multi-Step Problem** Three tennis balls fit tightly in a can as shown. Recall that the formula for the volume of a cylinder is \( V = \pi r^2 h \) and the formula for the volume of a sphere is \( V = \frac{4}{3} \pi r^3 \).

a. Write an expression for the height of the can, \( h \), in terms of \( r \). Rewrite the formula for the volume of a cylinder with \( r \) as the only variable.

b. Find the ratio of the volume of the three tennis balls to the volume of the can.

c. **Writing** Do you think using a cylindrical can is an efficient way of packaging tennis balls? Explain your reasoning.

12. **Multi-Step Problem** The length \( l \) and width \( w \) of a golden rectangle satisfy the equation \( \frac{l}{w} = \frac{l + w}{l} \). The ratio \( \frac{l}{w} \) is called the golden ratio. For centuries, golden rectangles have been known to be very pleasing to the human eye.

a. Rewrite the right side of the equation as a complex fraction by dividing each term of the numerator and denominator by \( w \).

b. Let \( g \) represent the golden ratio, so \( g = \frac{l}{w} \). Substitute \( g \) for each occurrence of \( \frac{l}{w} \) in the equation from part (a) and simplify the equation.

c. Solve the equation from part (b) for \( g \). (Hint: Use the quadratic formula.) Write an exact value and an approximate value for the golden ratio.

d. **Geometry Connection** Use a ruler or graph paper to draw an accurate golden rectangle of any size. Label the dimensions of your rectangle.
Cumulative Practice

for Chapters 1–9

Solve the equation for \( y \). (1.4)

1. \( 6x - 2y = 7 \)  
2. \( -\frac{3}{4}x - y = 9 \)  
3. \( \frac{1}{3}x + \frac{2}{5}y = 10 \)  
4. \( xy + 5x = -4 \)

Tell whether the lines are parallel, perpendicular, or neither. (2.2)

5. Line 1: through (2, 1) and (−6, 1)  
   Line 2: through (0, −3) and (−2, −3)  
6. Line 1: through (−1, 1) and (5, −1)  
   Line 2: through (2, 5) and (3, 2)

Graph the system of linear inequalities. (3.3)

7. \( y < x + 3 \)  
   \( y \geq -2x + 1 \)  
8. \( y < \frac{1}{2}x \)  
   \( y + 5 > \frac{1}{2}x \)  
9. \( 2x + 3y < 12 \)  
   \( x - 6y < 6 \)  
10. \( x \geq 0 \)  
    \( y \geq 0 \)  
    \( y \leq 4 \)

Perform the indicated operation, if possible. If not possible, state the reason. (4.1, 4.2)

11. \[
\begin{bmatrix}
10 & 3 \\
-6 & -1
\end{bmatrix}
- 
\begin{bmatrix}
-2 & 5 \\
6 & -3
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
1 & 0 & 6 \\
-3 & 5 & -2 \\
2 & 8 & -1
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
1 & -1 & -2 \\
4 & 3 & -5
\end{bmatrix}
\]

Plot the numbers in the same complex plane and find their absolute values. (5.4)

14. \( 1 + 4i \)  
15. \( -2 + i \)  
16. \( -i \)  
17. \( 6 \)  
18. \( -1 - 3i \)  
19. \( 3 - 5i \)

Find all the zeros of the polynomial function. (6.6, 6.7)

20. \( f(x) = x^3 + 2x^2 - 11x - 12 \)  
21. \( f(x) = x^3 - 5x^2 + 5x - 25 \)  
22. \( f(x) = x^4 - 81 \)

Simplify the expression. Assume all variables are positive. (6.1, 7.2, 8.3)

23. \( \frac{3x^5}{5y} \cdot \frac{xy}{2x^2} \)  
24. \( (-6x^{-2}y)^{-2} \)  
25. \( \sqrt[3]{16a^{4}b^{5}c} \)  
26. \( (9x^6)^{3/2} \)  
27. \( \left( \frac{1}{2}e^{-2} \right)^3 \)  
28. \( \frac{100e^{6x}}{24e^{4x}} \)

Evaluate the expression without using a calculator. (7.1, 8.4)

29. \( \sqrt[6]{64} \)  
30. \( -(100^{3/2}) \)  
31. \( 125^{-1/3} \)  
32. \( \log_2 \frac{1}{32} \)  
33. \( \log_7 \sqrt{7} \)  
34. \( \log 0.1 \)

Perform the indicated operation and state the domain. (7.3)

35. \( f - g; f(x) = 2x - 7, \ g(x) = x^2 - 20 \)  
36. \( f \cdot g; f(x) = 3x^{1/4}, \ g(x) = -x^{5/4} \)  
37. \( f(g(x)); f(x) = x - 10, \ g(x) = -2x^2 - 5 \)  
38. \( g(f(x)); f(x) = x + 6, \ g(x) = x^2 - 7x + 3 \)

Find the inverse of the function. (7.4, 8.4)

39. \( f(x) = \frac{1}{2}x - 6 \)  
40. \( f(x) = x^2 + 1, \ x \geq 0 \)  
41. \( f(x) = \log_5 \)  
42. \( f(x) = \ln 3x \)

Condense the expression. (8.5)

43. \( \log 3 + 2 \log x + 3 \log y \)  
44. \( \log_7 4 + \log_7 y - 2 \log_7 3 \)  
45. \( 2(\log x + \log y) \)
Graph the function. (7.5, 8.1–8.4, 8.8, 9.2, 9.3)

46. \( y = \sqrt{x + 12} \)  
47. \( y = 2x^{3/2} - 3 \)  
48. \( y = 3\left(\frac{4}{3}\right)^x \)  
49. \( y = 3\left(\frac{1}{2}\right)^x + 2 \)

50. \( y = e^x - 5 \)  
51. \( y = \log(x - 1) \)  
52. \( y = \ln x - 2 \)  
53. \( y = \frac{2}{1 + e^{-3x}} \)

54. \( y = \frac{5}{x - 2} - 1 \)  
55. \( y = \frac{x - 4}{2x + 1} \)  
56. \( y = \frac{13}{x^2 - 4} \)  
57. \( y = \frac{3x^2 + 5x - 2}{x - 1} \)

Solve the equation. Check each solution. (7.6, 8.6, 8.8, 9.6)

58. \( 2\sqrt{x + 5} = 18 \)  
59. \( \frac{1}{8}(x - 6)^{3/2} = 1 \)  
60. \( 2^x = 8^{x + 6} \)  
61. \( 4 \log_3 (-2x) = 10 \)

62. \( 2 \ln x + 5 = 7 \)  
63. \( \frac{5}{1 + 2e^{-x}} = 4 \)  
64. \( \frac{2x}{x + 4} = \frac{5}{2x - 3} \)  
65. \( \frac{1}{x - 2} - \frac{4}{x^2 - 4} = 5 \)

Write an exponential function of the form \( y = ab^x \) whose graph passes through the given points. (8.7)

66. \((1, 2), (3, 10)\)  
67. \((5, 5), (6, 10)\)  
68. \((2, 4), (4, 8)\)  
69. \((0.5, 1), (5, 12)\)

Write a power function of the form \( y = ax^b \) whose graph passes through the given points. (8.7)

70. \((1, 1), (3, 9)\)  
71. \((2, 3), (10, 12)\)  
72. \((4, 1), (8, 7)\)  
73. \((0.1, 1), (2, 2)\)

Perform the indicated operations. Simplify the result. (9.4, 9.5)

74. \( \frac{3x^2y}{x - 2} \cdot \frac{x^2 + x - 6}{3x - 6} + (x^2 - 4) \)  
75. \( \frac{6x}{3x + 1} + \frac{9x}{2x} - \frac{x + 1}{x - 1} \)

76. **FOOTBALL** The circumference of a standard football can vary from \( 20\frac{3}{4} \) inches to \( 21\frac{1}{4} \) inches around the middle and from \( 27\frac{3}{4} \) inches to \( 28\frac{1}{2} \) inches around the length of the football. Write two absolute value inequalities that describe the possible circumferences \( C_m \) and \( C_l \) of a football. (1.7)

77. **WISHING WELL** A stone is dropped into a deep wishing well. The water level of the well is 200 feet below the top of the well. After how many seconds will the stone hit the water? (5.3)

78. **HOME RUNS** In Exercises 78–80, use the following data set of number of home runs hit by each member of the Chicago Cubs in the 1998 baseball season. (7.7)

\[
8, 1, 17, 66, 1, 1, 8, 14, 8, 2, 0, 9, 23, 31, 0, 5, 7, 4, 0, 0, 2, 1, 0
\]

78. Find the mean, median, mode, range, and standard deviation of the data set.

79. Draw a box-and-whisker plot of the data set.

80. Make a frequency distribution of the data set. Use four intervals beginning with 1–10. Then draw a histogram of the data set.

**SCIENCE CONNECTION** In Exercises 81 and 82, use the following information.

The electrical force \( f \) (in newtons) between two charged particles varies jointly with the electric charges \( q_1 \) and \( q_2 \) (in coulombs) of the particles and inversely with the square of the distance \( r \) (in meters) between the particles. (9.1)

81. Write an equation relating \( f, q_1, q_2, r, \) and a constant \( k \).

82. Using \( k = 8,987,760,000 \), find the electrical force between a 2 coulomb charged particle and a 3 coulomb charged particle if the two particles are 2 meters apart.
Mathematical Models of Learning

OBJECTIVE  Graph data about learning time and fit a model to the data.

Materials: simple puzzle, timer

When you were first learning to read, a page in a children’s book might have taken you several minutes to work through. Now when reading a book to a small child, you can read an entire page easily. How did the time needed to read a page change as you got older and better at reading?

Many scientists study learning. Some study which environments seem to promote learning and which hinder it. Some study which parts of the brain are active when people learn a new skill and which are active when they practice old ones. Others explore how learning changes over time. In this project you will measure the time it takes to do a task as it becomes more familiar to you. You will then fit a model to your data.

INVESTIGATION

1. Find or create a simple task for someone to learn. The task should not take too long to complete and should gradually get easier with practice. Note, a brainteaser puzzle that is very difficult until you realize the trick and then is very easy will not show a gradual increase in performance. A task that is too easy will show only a slight variation in performance.

Possible tasks include: assembling a small puzzle, such as a 10–30 piece jigsaw puzzle; putting 20 index cards with words on them in alphabetical order; solving 15 arithmetic problems involving order of operations where the same problems are in a different order each time; arranging shapes to match a given pattern.

2. Choose a partner as a test subject. Explain to your partner how to do the task you chose. Administer the task to your partner and then measure the time your partner takes to complete the task. Record the data in a table like the one on the right. Repeat the process nine times.

3. Make a scatter plot of your data.

4. So far in this book you have studied linear, quadratic, polynomial, power, radical, exponential, logarithmic, and rational functions. Find a mathematical model that is one of these types of functions to fit your data.

<table>
<thead>
<tr>
<th>Task number</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
</tr>
<tr>
<td>3</td>
<td>?</td>
</tr>
<tr>
<td>4</td>
<td>?</td>
</tr>
<tr>
<td>5</td>
<td>?</td>
</tr>
<tr>
<td>6</td>
<td>?</td>
</tr>
<tr>
<td>7</td>
<td>?</td>
</tr>
<tr>
<td>8</td>
<td>?</td>
</tr>
<tr>
<td>9</td>
<td>?</td>
</tr>
<tr>
<td>10</td>
<td>?</td>
</tr>
</tbody>
</table>
PRESENT YOUR RESULTS

Write a report to present your results.

• Begin with a description of the learning task you used and explain how you chose it.
• Include your table of data, your graph, and your mathematical model.
• Explain how you chose the type of function to model your data.
• Explain how you obtained your model.
• Write a description of your data and what they show.

Extend your results.

• Recruit a second partner and repeat the experiment.
• Find a mathematical model to describe the learning time for the new data.
• Do you get the same or different results this time? Explain why you might expect similar or different results. (For example, you might obtain similar results because both subjects are juniors in high school who like word puzzles. You might obtain different results because one subject is several years younger than the other.)

EXTENSION

You have measured the times it takes to perform an increasingly familiar task. Now you will measure the times it takes to perform an increasingly complicated task.

Possible tasks include: assembling a puzzle cut out of paper with 2 pieces, then one with 4 pieces, then one with 8 pieces, and so on; arranging index cards with words on them in alphabetical order, first doing 5 cards, then 10 cards, then 15 cards, and so on; finding the ace of spades in a deck with 4 cards, then 8 cards, and so on.

Choose a task and recruit a volunteer to help you. Measure the times it takes your volunteer to complete the task 10 times at increasing levels of difficulty. Record the data in a table, make a scatter plot of the data, and then find a mathematical model to fit your data.